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LIE NILPOTENCE AND LIE SOLVABILITY IN GROUP ALGEBRAS

by

Benjamin Baird

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Master of Science

in

Mathematics

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Lie Nilpotence and Lie Solvability in Group Algebras** submitted by **Benjamin Baird** in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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ABSTRACT

In this thesis we will begin by exploring conditions under which a group algebra FG is Lie nilpotent and conditions under which it is Lie solvable. We will then turn our attention to the symmetric and skew symmetric elements, and look for conditions under which each of these subsets is Lie nilpotent and conditions under which they are Lie solvable. It is interesting to note that in many situations if the subset of symmetric or skew symmetric elements have this condition, then indeed the whole group algebra also must satisfy the condition.

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Chapter 1

Background

1.1 Introduction

In this thesis we will be studying group algebras which have certain special conditions put on them.

Definition 1.1. Given any field F and group G let FG denote the set of all formal linear combinations $\alpha = \sum_{g \in G}(\alpha_g g)$ with $\alpha_g \in F$, $g \in G$, and $\alpha_g \neq 0$ for only a finite subset of G. Given 2 elements of FG we define an addition by $\sum_{g \in G}(\alpha_g g) + \sum_{g \in G}(\beta_g g) = \sum(\alpha_g + \beta_g)g$ and a multiplication by $(\sum_{g \in G}(\alpha_g g))(\sum_{h \in G}(\beta_h h)) = \sum_{g,h \in G}(\alpha_g \beta_h)gh$. FG is called a group algebra.

It should be noted here that the above definition insures that FG is actually an algebra over F. Generally when we study any algebraic structure we begin by considering the case when the structure is commutative. As we generalize to non-commutative cases we often put restrictions on the structure so that it is "almost" commutative. In the case of algebras one way of doing this is to study algebras which satisfy polynomial identities.

Definition 1.2. A group algebra FG is said to satisfy a polynomial identity if there exists a non-zero polynomial $f(x_1, ..., n_n) \in F\langle x_1, ..., x_n \rangle$ such that $f(\alpha_1, ..., \alpha_n) = 0$ for all $\alpha_1, ..., \alpha_n \in FG$.

Notice that a group algebra is commutative if and only if it satisfies the polynomial identity $xy - yx$. We will be studying group algebras that satisfy some particular polynomial identities in which case they will be called Lie nilpotent or Lie solvable.

Definition 1.3. A group algebra FG is Lie nilpotent of index n if $[r_1, ..., r_n]$ is a polynomial identity for FG where $[r_1, r_2] = r_1r_2 - r_2r_1$ and

$[r_1, ..., r_i] = [[r_1, ..., r_{i-1}], r_i]$ for $i \geq 3$.

Definition 1.4. A group algebra FG is Lie solvable if $\delta^n(r_1, ..., r_{2^n})$ is a polynomial identity for FG for some n, where $\delta^1(r_1, r_2) = [r_1, r_2]$ and

$\delta^i(r_1, ..., r_{2^i}) = [\delta^{i-1}(r_1, ..., r_{2^{i-1}}), \delta^{i-1}(r_{2^{i-1}+1}, ..., r_{2^i})]$.

We should note right away that it is immediate from the definitions that if FG is Lie nilpotent of index n, then FG satisfies $\delta^n(r_1, ..., r_{2^n})$. Thus Lie nilpotence of a group algebra implies Lie solvability.

Lie nilpotent and Lie solvable group algebras have been studied extensively, and a full characterization of them has been obtained as will be seen at the end of this chapter. Despite this characterization other questions still arise. In particular any group algebra has a natural involution by extending the map $* : G \to G$, with $g^* = g^{-1}$ to the group algebra, this will be formally defined in chapter 2. This involution gives rise to the study of subsets of FG which have the special property that $x^* = x$ or $x^* = -x$, we call these subsets the symmetric or skew symmetric elements respectively. Chapters 2 and 3 will focus on the question of when these subsets of FG are Lie nilpotent or Lie solvable. In particular it is interesting to notice that in some cases the Lie nilpotence or Lie solvability of one of these subsets implies the Lie nilpotence or Lie solvability of FG.

The question of when these subsets are Lie nilpotent was addressed and answered with two extra assumptions in [7]. These assumptions were that the field F has characteristic different from 2, and that the group G contained no elements of order 2. Under these conditions the authors were able to show that if either of these subsets was Lie nilpotent then the whole group algebra

FG was actually Lie nilpotent. More recently in [12] the author was able to characterize when the symmetric elements are Lie nilpotent in a group algebra with characteristic of the field different from 2, thus removing the condition that G contained no elements of order 2. The question of characterizing FG when the skew symmetric elements were Lie nilpotent with characteristic of F different from 2 still remained unanswered though. This was solved earlier this year in [5]. The question of characterizing FG when these sets are Lie nilpotent and characteristic of F is 2 remains open. I explore this question in the third section of chapter 2 and some partial results can be found there. A full characterization of this case would complete the characterization of group algebras in which either the symmetric or skew symmetric elements are Lie nilpotent.

The other question I alluded to earlier was characterizing FG when the symmetric or skew symmetric elements are Lie solvable. I begin to explore this question in the case where the characteristic of F is not 2 and G contains no elements of order 2. These results can be found with the results about Lie nilpotence under the same conditions in section 2.1. Further exploration into this question has yet to be done.

Before we begin to discuss the particular questions with which this thesis deals we are going to mention a few definitions and results that are basic to almost any study of group algebras. In the study of any rings, or algebras in particular, ideals generally end up playing a very important role in determining the structure of the ring. The augmentation ideals are among the most important ideals in studying group algebras.

Definition 1.5. ([15] Proposition 3.3.4.) Given a group algebra FG and a normal subgroup N of G we define the augmentation ideal $\Delta(G, N)$ to be the kernel of natural map $FG \to F(G/N)$.

In particular, G is always normal in itself, and we often write $\Delta(G)$ for $\Delta(G, G)$.

Proposition 1.1. *([15] Proposition 3.3.3.) Let F be a field and G be a group*

with normal subgroup N and let T be a transversal of N in G. The set $\{t(n-1)|t \in T, n \in N\}$ is an F basis for $\Delta(G,N)$.

Proof. Let $f : FG \to F(G/N)$ be the natural map. Clearly $f(t(n-1)) = \overline{t}(\overline{n}-1) = \overline{t}(1-1) = 0$. This set is also clearly linearly independent because fixing t and n, the group element tn only shows up in one term of the set, and group elements are linearly independent in FG. Now suppose $\alpha = \sum_G(\alpha_g g) \in ker(f)$. We can rewrite alpha uniquely as $\alpha = \sum_{t\in T,n\in N}(\alpha_{tn}tn)$, by the definition of transversal. Thus $f(\alpha) = \sum_{t\in T,n\in N}(\alpha_{tn}\overline{tn}) = \sum_{t\in T,n\in N}(\alpha_{tn}\overline{t} = 0$. This implies that for any $t \in T$ we have $\sum_N(\alpha_{tn}t) = 0$, therefore $\alpha = \alpha - \sum_N(\alpha_{tn}t)$. We can do this for each $t \in T$ giving $\alpha = \sum_{t\in T,n\in N}(\alpha_{tn}tn - \alpha_{tn}t) = \sum_{t\in T,n\in N}(\alpha_{tn})(t)(n-1)$. Thus the set given generates $\Delta(G,N)$.

Again in the special case when $N = G$ we have the F basis $\{g-1|g \in G$. □

Before moving on the assumption that FG is Lie nilpotent or Lie solvable, we will prove some background results about group algebras that will be useful throughout the rest of the thesis.

1.2 Background lemmas

Lemma 1.1. *If a group algebra FG satisfies a polynomial identity of degree n, then it satisfies a polynomial identity of the form $g = \sum_{\sigma\in S_n}(a_\sigma x_{\sigma(1)}...x_{\sigma(n)})$ with $a_1 = 1$.*

Proof. Assume FG satisfies a polynomial identity f of degree n, and recall that $f \in F\langle x_1,...,x_m\rangle$. If there exists some x_i that appears in some but not all terms of f we can write $f' = f(x_1,...,x_{i-1},0,...,x_m)$. Then f' is a non-zero polynomial identity for FG, and f' is a function of fewer variables than f. We can continue this way until we arrive at a polynomial identity g for FG with $deg(g) \leq n$ and g has the property that each variable that appears in g appears in every term of g. Thus with a possible relabelling of variables we may assume $g \in F\langle x_1,...,x_k\rangle$ with $k \leq n$ because $deg(g) \leq n$. Let B be the set of polynomial identities h for FG with $deg(h) \leq n$ and for which every variable

which appears in h appears in every term of h. The above arguments show that the set B is non-empty, hence we can choose $h \in B$ with the property that h is a function of a maximal number of variables.

Suppose some h is not linear in all its variables then h is not a function of all variables $x_1, ..., x_n$, because if some term contains n variable and is not linear in all those variables it has degree greater than n. Without loss of generality we can assume $h \in F\langle x_1, ..., x_k\rangle$ with $k < n$ and that h is not linear in x_1. Now set $l(x_1, ..., x_{k+1}) = h(x_1 + x_{k+1}, x_2, ...) - h(x_1, x_2, ..., x_k) - h(x_{k+1}, x_2, ..., x_k)$. This is also a polynomial identity for FG and $l \neq 0$ because otherwise h would be linear in x_1. Each variable in l appears in every term by construction, for if we write $h = h_1 + h_2$ with h_1 equal to the sum of all monomials in h which are not linear in x_1 then we have $l = h_1(x_1 + x_{k+1}, x_2, ..., x_k) + h_2(x_1 + x_{k+1}, x_2, ..., x_k) - h_1(x_1, x_2, ..., x_k) - h_1(x_{k+1}, x_2, ...x_k) - h_2(x_1, x_2, ..., x_k) - h_2(x_{k+1}, x_2, ...x_k) = h_1(x_1 + x_{k+1}, x_2, ..., x_k) - h_1(x_1, x_2, ..., x_k) - h_1(x_{k+1}, x_2, ..., x_k)$ and the terms in $h_1(x_1 + x_{k+1}, x_2, ...)$ which contain x_1 but not x_{k+1} are exactly $h_1(x_1, x_2, ..., x_k)$ and similarly the terms which contain x_n but not x_1 are exactly $h_1(x_{k+1}, x_2, ..., x_k)$, leaving only terms containing both. Also each term of l will contain all other variables that h contained, and $deg(l) \leq deg(h) \leq n$, hence $l \in B$. But l is a function of more variables than h contrary to h being maximal in B with this property. Thus we can conclude that h is linear in all its variables, and hence if h is a function of k variables, it has degree k. Suppose $k < n$, then we set $r(x_1, ..., x_{k+1}) = x_{k+1}h$ and find that $r \in B$ and r is a function of more variables than h. This contradiction leads us to the conclusion that h must contain all variables $x_1, ..., x_n$ in each term, and thus $deg(h) = n$. Therefore we have that

$h = \sum_{\sigma \in S_n}(a_\sigma x_{\sigma(1)}...x_{\sigma(n)})$ with $a_\sigma \neq 0$ for some $\sigma \in S_n$. With a possible relabelling of indices, we can assume that $a_1 \neq 0$, and hence we can further assume that $a_1 = 1$ by replacing h with $a_1^{-1}h$. □

Lemma 1.2. *If a group algebra FG satisfies a polynomial identity, then $[G : \phi_k(G)] < (k+1)!$ for some k, where $\phi_k(G) = \{x | [G : \mathbf{C}(g)] \leq k\}$.*

Proof. We can assume the polynomial identity for FG is of the form $g = x_1...x_n + \sum_{1 \neq \sigma \in S_n}(a_\sigma x_{\sigma(1)}...x_{\sigma(n)})$ by lemma 1.1.

Now we will define $f_i(x_i, ..., x_n)$ by

$f = x_1...x_{i-1}f_i +$ (*terms in* f *that do not begin with* $x_1...x_{i-1}$). We should note here that $f_1 = f$ and since the only term that starts with $x_1...x_{n-1}$ is $x_1...x_n$ $f_n = x_n$, and $f_j = x_j f_{j+1} +$ (*terms in* f_j *that do not start with* x_j).

We will let $k = (n!)^2$ let $\mathcal{M}_j$ be the set of all linear monomials in $F\langle x_j, ..., x_n\rangle$ for $j \geq 2$ and let $\mathcal{M}_1$ be empty. Since $f_1(x_1, ...x_n) = 0$ for any $x_1, ...x_n \in$G we can conclude that there exists some $0 \leq i \leq n$ so that for any $j \leq i$ and $x_j, ..., x_n \in G$, either $f_j(x_j, ...x_n) = 0$ or $\mu(x_j, ...x_n) \in \phi_k(G)$ for some $\mu \in \mathcal{M}_j$. Now since $f_n(x) = x \neq 0$ for any $x \in G$, and since $G \neq \phi_k(G)$ we can conclude that $i < n$. Thus with i chosen to be maximal, we can conclude that there exists $x_{i+1}, ..., x_n \in G$ with $f_{i+1}(x_{i+1}, ..., x_n) \neq 0$ and $\mu(x_{i+1}, ..., x_n) \notin \phi_k(G)$ for any $\mu \in \mathcal{M}_{i+1}$.

Now fix this $x_{i+1}, ..., x_n \in G$ and set $\mathcal{T} = \mathcal{M}_i \setminus \mathcal{M}_{i+1}$. For any $\mu \in \mathcal{T}$ we can write $\mu = \mu' x_i \mu''$ with $\mu', \mu'' \in \mathcal{M}_{i+1}$. Thus we can define $T = \bigcup_{\mu \in \mathcal{T}} \mu'(x_{i+1}, ..., x_n)^{-1}\phi_k(G)\mu''(x_{i+1}, ..., x_n)^{-1}$. So for any $\mu \in \mathcal{T}$ we have $\mu(x, x_{i+1}, ..., x_n) \in \phi_k(G)$ implies $x \in T$. Thus for $x \in G \setminus T$ we must have $f_i(x, x_{i+1}, ..., x_n) = 0$ by assumption.

We can write $f_i(x_i, ...x_n) = x_i f_{i+1}(x_{i+1}, ..., x_n) + \sum_{k=1..t}(r_k \mu'_k x_i \mu''_k)$ with $r_k \in F$ and $\mu'_k, \mu''_k \in \mathcal{M}_{i+1}$ and $t < n!$ because the number of terms in f is less or equal to $n!$, thus $x f_{i+1}(x_{i+1}, ..., x_n) = \sum_{k=1..t}(-r_k \mu'_k x \mu''_k)$ for all $x \in G \setminus T$.

Note that $\mu'_l, \mu''_l \in G$ for each l. If μ'_l is conjugate to $v\mu''^{-1}_j$ for some $v \in supp(f_{i+1}(x_{i+1}, ..., x_n)$ we can write $h_j^{-1}\mu'_k h_j = v\mu''^{-1}_j$ for some $h_j \in G$, and thus we can define $S = \bigcup_{l,j} \mathbf{C}_G(\mu'_l)h_j$ where l, j run through all combinations where μ'_l is conjugate to $v\mu''^{-1}_j$ for some $v \in supp(f_{i+1}(x_{i+1}, ..., x_n)$. Now given any $x \in G \setminus T$ we have $f_{i+1}(x_{i+1}, ..., x_n) = x^{-1}\sum_{l=1..t}(-r_l \mu'_l x \mu''_l)$ and thus for $v \in supp(f_{i+1}(x_{i+1}, ..., x_n)$ we have $v = x^{-1}\mu'_l x \mu''_l$ for some l, and hence $v\mu''^{-1}_l = x^{-1}\mu'_l x = h_j^{-1}\mu'_l h_j$. So we have that $\mu'_l x h_j^{-1} = x h_j^{-1}\mu'_l$ hence $x h_j^{-1} \in \mathbf{C}_G(\mu'_l)$ and thus $x \in S$.

So for any $x \in G \setminus T$ we have $x \in S$ thus $G = T \bigcup S$. Notice that

$[G : \mathbf{C}_G(\mu_l')] > k$ because $\mu_l \notin \phi_k(G)$, and also notice that in defining S there are at most $t^2 < (n!)^2 = k$ cosets because $1 \leq l, j \leq t$ and thus $G \neq S$ by [14] lemma 5.2.2 and hence $[G : T] \leq (t^2+1)! \leq k!$ by [14] lemma 5.2.1. Now $[G : \phi_k(G)] \leq |\mathcal{T}|[G : T]$ by the definition of T, but since $\mathcal{T} \subseteq \mathcal{M}_2$ and simply counting we find there are $(n-1)!$ monomials of degree $n-1$ in $\mathcal{M}_2$ and fewer monomials of each degree $j < n-1$, thus $|\mathcal{M}_2| < n(n-1)! < n!$. Thus we have a bound $[G : \phi_k(G)] \leq |\mathcal{T}|[G : T] \leq (n!)(k!) < (k+1)!$. □

Theorem 1.1. *If a group algebra FG satisfies a polynomial identity, then $[G : \phi(G)] < \infty$ and $|\phi(G)'| < \infty$ where $\phi(G) = \{x | [G : C(g)] < \infty\}$ is the FC-subgroup of G.*

Proof. Simply noting that $\phi_k(G) \subseteq \phi(G)$ we can infer that $(k+1)! > [G : \phi_k(G)] \geq [G : \phi(G)]$ for some k by the last theorem. Thus it remains to show that $|\phi(G)'| < \infty$. Suppose $a, b \in \phi_k(G)$, and $g \in G$. If $ag \in \phi(G)$ then $ba^{-1}ag = bg \in \phi(G)$, thus $\phi(G) \bigcap \phi_k(G)g \neq \emptyset$ if and only if $\phi_k(G)g \subseteq \phi(G)$. Thus we can write $\phi(G) = \phi_k(G)g_1 \bigcup ... \bigcup \phi_k(G)g_n$, we know this is a finite union because $[\phi(G) : \phi_k(G)] \leq [G : \phi_k(G)] < \infty$. Also notice that $g_i \in \phi_k(G)g_i \subseteq \phi(G)$ thus we can let $u = max\{[G : \mathcal{C}_G(g_i)] | i = 1..n\} < \infty$ because this is a finite set. Now for any $g \in \phi(G)$ we have $g = ag_i$ for some i and some $a \in \phi_k(G)$. Now any conjugate $h^{-1}gh$ of g, can be written $h^{-1}gh = h^{-1}ahh^{-1}g_ih$, as a product of conjugates of a and g_i, thus the number of distinct conjugates of g must be less or equal to the number of conjugates of a times the number of conjugates of g_i. In other words $[G : \mathcal{C}_G(g)] \leq [G : \mathcal{C}_G(a)][G : \mathcal{C}_G(g_i)] \leq ku$. Thus we have shown that $\phi_{ku}(G) = \phi(G)$. This implies that $|(\phi(G))'| < \infty$. □

The previous result can be improved as in [14] theorem 5.2.14, to $[G : \phi(G)] \leq n/2$ where n is the degree of the polynomial identity.

Definition 1.6. A group algebra FG is said to be semiprime if it has no non-zero nilpotent ideals.

Theorem 1.2. *Let F be a field with $char(F) = p$ and G a group. Then FG is semiprime if $p = 0$ or $p > 0$ and G contains no p-elements.*

Proof. Assume $p = 0$. Suppose I is a nilpotent ideal of FG and $\alpha \in I$ with $\alpha = \sum \alpha_{g_i} g_i$. We can define $E = \mathbb{Q}(\alpha_{g_1}, ..., \alpha_{g_n})$ an extension of the rational numbers, and then E can be embedded in the complex numbers. Fixing this embedding ϕ, we can pick a field K with, $\mathbb{C} \supseteq K \supseteq \phi(E)$ and K closed under complex conjugation. Now $I' = I \bigcap EG$, is a nilpotent ideal of EG and $KG = EG \otimes_E K$, thus $I' \otimes_E K$ is a nil ideal of KG. Now if $\gamma = \sum \gamma_g g \in I' \otimes_E K$ and we define $\gamma^* = \sum \overline{\gamma_g} g^{-1}$ and $\delta = \gamma\gamma^*$ then $\delta_1 = \sum \gamma_g \overline{\gamma_g} = \sum |\gamma_g|^2 \neq 0$ if $\gamma \neq 0$ therefore $\delta \neq 0$ and $\delta^* = (\gamma^*)^* \gamma^* = \delta$. Now we have $\delta \in I' \otimes_E K$ and thus δ is nilpotent and $\delta = \delta^*$. But if δ is nilpotent, then $\delta^{2^n} = 0 = \delta^{2^{n-1}} (\delta^{2^{n-1}})^*$. This implies that $\delta^{2^{n-1}} = 0$ by the same argument used to show that $\delta \neq 0$. Repeating this process clearly shows after n steps that $\delta = 0$, and thus $\gamma = 0$. Hence we conclude that $I' \otimes_E K = 0$ and therefore $I' = 0$ and thus $I = 0$. This implies FG has no non-zero nilpotent ideals, and hence FG is semiprime.

For the $p > 0$ case we will do a few preliminary calculations. Thus assume that $p > 0$. Given $x, y \in FG$ we have $(x + y)^p = x^p + y^p + \sum(z_1 ... z_p)$. where $z_i = x, y$ and the sum runs through all possible combinations except when all terms are either x or y. We can break the sum up into disjoint sums, each containing some $z_1 ... z_p$ along with all its cyclic permutations $z_i ... z_p z_1 ... z_{i-1}$, call an arbitrary such sum δ. Each such permutation can be written as $\alpha\beta$ with $\beta\alpha = z_1 ... z_p$. Thus each term in the sum is of the form $\beta\alpha + [\alpha, \beta]$. Hence we can write $\delta = p(z_1 ... z_p)(mod[FG, FG]) = 0(mod[FG, FG])$. Thus $(x+y)^p = x^p + y^p (mod[FG, FG])$. Now the following calculation shows that for $\gamma \in [FG, FG]$ we have $\gamma^p \in FG$: $(xy - yx)^p = (xy)^p - (yx)^p (mod[FG, FG]) = [x, (yx)^{p-1} y](mod[FG, FG]) = 0(mod[FG, FG])$. Thus inductively we assume that $(x+y)^{p^{n-1}} = x^{p^{n-1}} + y^{p^{n-1}} + \gamma$ with $\gamma \in [FG, FG]$, and calculating we have $(x+y)^{p^n} = (x^{p^{n-1}} + y^{p^{n-1}} + \gamma)^p = (x^{p^{n-1}} + y^{p^{n-1}})^p + \gamma^p + \gamma' = x^{p^n} + y^{p^n} + \gamma'' + \gamma^p + \gamma'$, with $\gamma^p, \gamma', \gamma'' \in [FG, FG]$. Thus $(x+y)^{p^n} = x^{p^n} + y^{p^n} (mod[FG, FG])$. Clearly then for any $\alpha = \sum \alpha_g g$ we have $\alpha^{p^n} = \sum \alpha^{p^n} g^{p^n} (mod[FG, FG])$.

Now suppose G contains no p-elements and suppose I is a nilpotent ideal of

FG. Given $\alpha \in I$, we can assume, by multiplying by some group element, that $1 \in supp(\alpha)$. By assumption $\alpha^{p^n} = 0$ for some n. Thus $0 = (\sum_{g \in g} \alpha_g g)^{p^n} = \sum_{g \in g} \alpha_g^{p^n} g^{p^n} + \gamma = \sum_{g^{p^n}=1} \alpha_g^{p^n} + \sum_{g^{p^n} \neq 1} \alpha_g^{p^n} g^{p^n} + \gamma$. for some $\gamma \in [FG, FG]$ by the previous arguments. Now $1 \notin supp(\gamma)$ thus the first sum must be zero. This implies that $\alpha_1^{p^n} = 0$ because G has no p-elements. This is a contradiction as $\alpha_1 \neq 0$ by assumption. Thus FG contains no nilpotent ideals. □

Definition 1.7. For simplicity of notation we will write $[x, y, ..., y] = [x, {}_r\, y]$ where we have r copies of y in the left hand Lie bracket and $x, y \in FG$ and similarly for sets we write $[X, Y, ..., Y] = [X, {}_r\, Y]$ when we have r copies of Y in the left hand Lie bracket and $X, Y \subseteq FG$.

Lemma 1.3. *Suppose char(F)=p, then for any m, and any $x, y \in FG$ $[x, {}_{p^m}\, y] = [x, y^{p^m}]$.*

Proof. $[x, y] = xy - yx = (r_y - l_y)(x)$ where r_y and l_y are the operators for right and left multiplication by y respectively. Inductively assume that $[x, {}_k\, y] = (r_y - l_y)^k(x)$ then $[x, {}_{k+1}\, y] = [[x, {}_k\, y], y] = [(r_y - l_y)^k(x), y] = (r_y - l_y)^{k+1}(x)$. Thus $[x, {}_{p^m}\, y] = (r_y - l_y)^{p^m}(x) = (r_y^{p^m} - l_y^{p^m})(x)$ because right and left multiplication commute as operators. Thus $[x, {}_{p^m}\, y] = (r_y^{p^m} - l_y^{p^m})(x) = xy^{p^m} - y^{p^m}x = [x, y^{p^m}]$. □

Lemma 1.4. *Let R be a commutative ring with $char(R) = p^n$ for some prime p, and G be a group. Then the augmentation ideal $\Delta(G)$ is a nilpotent ideal if and only if G is a finite p-group.*

Proof. Recall from proposition 1.1 that $\Delta(G)$ can be described as having the R basis $\{g - 1 | g \in G\}$. If $|G| = p$, then $(g-1)^p = g^p - 1 + p(\beta) = p(\beta)$ for some $\beta \in RG$. Thus for $\alpha \in \Delta(G)$ we have $\alpha^p = p\beta$ for some $\beta \in RG$ and hence $\alpha^{np} = (p\beta)^n = p^n\beta^n = 0$. If the order of G is p^m for some $m > 1$ then G has a central subgroup $N = \langle x \rangle$ of order p. Thus inductively we can assume that $(\Delta(G/N))^k = 0$. This implies that $(\Delta(G))^k \subseteq \Delta(G, N) \subseteq (x-1)RG$ the second inclusion here is clear because the R-basis for $\Delta(G, N)$ from proposition 1.1 is included in the ideal $(x-1)RG$.

Conversely assume that $\Delta(G)$ is nilpotent. Then we have $(\Delta(G))^n = 0$ and $(\Delta(G))^{n-1} \neq 0$ for some n. Choosing some non-zero $\delta \in (\Delta(G))^{n-1}$ we have that $(g-1)\delta = 0$ for all $g \in G$ thus $g\delta = \delta$. Therefore if $h \in supp(\delta)$ then $gh \in supp(\delta)$ for all $g \in G$ hence $|G| = |Gh| = |supp(\delta)| < \infty$. Now $R \supseteq \mathbb{Z}/p^n\mathbb{Z}$ and a homomorphism exists from $\mathbb{Z}/p^n\mathbb{Z}$ onto $F = \mathbb{Z}/p\mathbb{Z}$ which implies that FG has a nilpotent augmentation ideal. Now choose m so that $p^m > n$ then $(g-1)^{p^m} = g^{p^m} - 1 = 0$ in FG, but this implies g is a p-element, thus G is a finite p-group. □

Definition 1.8. For $a, b \in G$ we define $(a,b) = a^{-1}b^{-1}ab$ and for $A, B \subseteq G$, $(A,B) = \{(a,b)|a \in A, b \in B\}$. Recursively we define $(x_1, ..., x_n) = ((x_1, ..., x_{n-1}), x_n)$, for $x_1, ..., x_n \in G$ and similarly $(A_1, ..., A_n) = \{(x_1, ..., x_n)|x_i \in A_i\}$ for $A_i \subseteq G$.

Again for notational simplicity we will write $(A, B, ..., B) = (A, {}_r B)$ when the left bracket has r copies of B.

Lemma 1.5. *Suppose A is a normal abelian subgroup of G, with finite exponent p^n, and suppose that G acting by conjugation is a finite p-group of automorphisms on A. Then $(A, {}_r G) = 1$ for some r.*

Proof. If we consider the subgroup A as a $(\mathbb{Z}/p^n\mathbb{Z})G - module$ with $a^g = g^{-1}ag$ for $a \in A$ and $g \in G$, thus for $\alpha = \sum \alpha_g g \in (\mathbb{Z}/p^n\mathbb{Z})G$ we have $a^\alpha = \prod (a^{\alpha_g})^g$. So if we calculate $(a,g) = a^{-1}g^{-1}ag = a^{-1}a^g = a^{g-1}$ we see that $(A,G) \subseteq A^{\Delta(G)}$. Inductively we can see assume that $(A, {}_{r-1} G) \subseteq A^{(\Delta(G))^{r-1}}$, and considering $a \in A^{(\Delta(G))^{r-1}}$ we see that $(a,g) = a^{g-1}$ thus for any n we have $(A, {}_n G) \subseteq A^{(\Delta(G))^n}$.

Noting that $G/\mathcal{C}(A)$ is a finite p-group because G acts as a finite p-group of automorphisms when acting by conjugation, thus we can conclude that $\Delta(G/\mathcal{C}(A))$ is nilpotent by the previous lemma. Thus for some r we have $(\Delta(G/\mathcal{C}(A)))^r = 0$ and hence $(\Delta(G))^r \subseteq (\Delta(G, \mathcal{C}(A)))$, thus $(A, {}_r G) \subseteq A^{(\Delta(G))^r} \subseteq A^{(\Delta(G, \mathcal{C}(A)))}$. But by considering any generator $g(h-1)$ of $(\Delta(G, \mathcal{C}(A)))$ we obtain $a^{g(h-1)} = (a^{(h-1)})^g = (h^{-1}aha^{-1})^g = 1^g = 1$ because $h \in \mathcal{C}(A)$. This implies that $(A, {}_r G) = 1$. □

1.3 Lie Nilpotence and Lie Solvability

We now have many of the basic theorems about group algebras satisfying polynomial identities. These will be used many times in this section as we characterize group algebras FG which are Lie nilpotent or Lie solvable.

Lemma 1.6. *Let F be a field with $char(F) = p$. Suppose that S is a commutative F-algebra, with or without 1, which is not nilpotent. If*

(i) $M_n(S)$ is Lie nilpotent then n=1, and

(ii) $M_n(S)$ is Lie solvable then n=1 or n=2 and p=2.

Proof. Let $T_m(S)$ be the set of trace 0 matrices and s_{ij} be the matrix with s in the i,j position and 0 elsewhere, for any $s \in S$. If $m \geq 3$ and $i \neq j$, choose $l \neq i,j$, then for any $r, s \in S$ we have $[s_{ij}, r_{jj} - r_{ll})] = sr_{ij}$ and $[s_{ij}, r_{ji}] = sr_{ii} - sr_{jj}$. Since the matrices on the right hand side of these equations span $T_m(S^2)$ where S^2 is the space spanned by $\{sr | s, r \in S\}$ and those on the right hand side are in $T_m(S)$ we can conclude that $[T_m(S), T_m(S)] \supseteq T_m(S^2)$. In the same way if we recall δ in the definition of Lie solvable, $\delta^n(T_m(S)) = [\delta^{n-1}(T_m(S)), \delta^{n-1}(T_m(S)]$. Inductively assume that $\delta^{n-1}(T_m(S)) \supseteq T_m(S^{2^{n-1}})$ therefore $\delta^n(T_m(S)) \supseteq [T_m(S^{2^{n-1}}), T_m(S^{2^{n-1}})]$. Now looking at the above equalities again with $s, r \in S^{2^{n-1}}$ proves that $\delta^n(T_m(S)) \supseteq T_m(S^{2^n})$. Also we get that $[T_m(S), {}_n M_m(S)] \supseteq [T_m(S), {}_n T_m(S)] \supseteq T_m(S^n)$.

Similarly if $m = 2$ we have $[s_{ij}, r_{jj}] = sr_{ij}$ and $[s_{ij}, r_{ji}] = sr_{ii} - sr_{jj}$) thus $[T_m(S), M_m(S)] \supseteq T_m(S^2)$. Once again from this we can do induction to show that $[T_m(S), {}_r M_m(S)] \supseteq T_m(S^r)$

Finally if $char(F) \neq 2$ and $m = 2$ then $[s_{ij}, r_{jj} - r_{ii}] = 2sr_{ij}$ and $[s_{ij}, r_{ji}] = sr_{ii} - sr_{jj}$. Thus $[T_m(S), T_m(S)] \supseteq T_m(S^2)$. By induction we have $\delta^n(T_m(S)) \supseteq T_m(S^{2^n})$.

Now if $m \geq 2$ and $M_m(S)$ is Lie nilpotent then $[M_m(S), {}_r M_m(S)] = 0$ for some r. Then $0 = [M_m(S), {}_r M_m(S)] \supseteq [T_m(S), {}_r M_m(S)] \supseteq T_m(S^r)$ and thus $S^r = 0$, this is a contradiction, as S is not nilpotent by assumption, therefore $m = 1$.

If $M_m(S)$ is Lie solvable and $m \geq 3$ or $m = 2$ and $p \neq 2$ then $0 = \delta^n(M_m(S)) \supseteq \delta^n(T_m(S)) \supseteq T_m(S^{2^n})$, and thus S is nilpotent, and this is a contradiction. □

The next theorem will be needed in chapter 2, but is related to the previous lemma, and so will be proved here. Several well known results will be needed in the proof, and I will state them in the following remarks without proof.

Remark 1.1. ([14] corollary 5.4.11.) Let R be a prime algebra satisfying a polynomial identity of degree n. Then R has a classical ring of quotients $E = \mathcal{Q}_c(R)$ that is isomorphic to $M_t(D)$ for some division algebra D. Furthermore, $dim_{Z(E)}(E) = m^2$ for some $m \leq n/2$.

Remark 1.2. ([10] Corollary to theorem.) 4.2.1. If D is a division algebra finite dimensional over its center F and K is a maximal subfield of D then $D \otimes_F K \simeq M_n(K)$ when $n = [D : K]$.

Theorem 1.3. *Assume an F algebra R is prime, and $char(F) \neq 2$. If R is Lie solvable then R is commutative.*

Proof. Since R is Lie solvable, it satisfies a polynomial identity, and thus $R \subseteq E \simeq M_t(D)$ where D is a division algebra and E is a classical ring of quotients of R by remark 1.1. E is Lie solvable because it is a ring of quotients of R, and thus $M_t(D)$ is Lie solvable. Now let $Z(E)$ be the center of E, then $dim_{Z(E)}E = m^2$, thus $m^2 = dim_{Z(M_t(D))}(Z(M_t(D))) = t^2 dim_{Z(D)}(D)$ and therefore $dim_{Z(D)}(D) = k^2$ for some k. Now we tensor D with a maximal subfield $K \supseteq Z(D)$ of D. Therefore we obtain $D \otimes_{Z(D)} K \simeq M_n(K)$ with $n = [D : F] \leq k^2$ by remark 1.2, and since this is a Lie solvable matrix ring over a field K with $char(K) \neq 2$ we find that $n = 1$ by lemma 1.6. This implies that $D \otimes_{Z(D)} K \simeq K$ and therefore $D = K$ thus D is a field. Now again we have $M_t(D)$ is Lie solvable and D is a field with $char(D) \neq 2$ thus $t = 1$ by lemma 1.6. This implies that $R \subseteq E \simeq D$ and thus R is commutative. □

Lemma 1.7. *Let $char(F) = p$. If FG is Lie nilpotent then G is an FC group. If FG is Lie solvable then either G is an FC group or the FC subgroup $\phi(G)$ has index 2 and $p = 2$.*

Proof. Assume one of the hypotheses of the lemma holds but $p \neq 0$, then $[G : \phi(G)] < \infty$ and $|\phi(G)'| < \infty$ by theorem 1.1. Let $\overline{G} = G/\phi(G)'$. Now for $y \in \mathcal{C}_G(x)$ we have $\overline{yx} = \overline{xy}$ thus $\overline{y} \in \mathcal{C}_{\overline{G}}(\overline{x})$ hence if $[G : \mathcal{C}_G(x)] < \infty$ then $[\overline{G} : \mathcal{C}_{\overline{G}}(\overline{x})] < \infty$, thus $\phi(G)/\phi(G)' \supseteq \phi(\overline{G})$. Also if $[\overline{G} : \mathcal{C}_{\overline{G}}(\overline{x})] = n$ then $[G : \mathcal{C}_G(x)] \leq |\phi(G)'|n < \infty$, thus $\phi(G)/\phi(G)' = \phi(\overline{G})$. Therefore $[G : \phi(G)] = [\overline{G} : \phi(\overline{G})]$, and $F\overline{G}$ satisfies the same polynomial identity as FG.

Now we need to show that $[G : \phi(G)] = 1$ or $[G : \phi(G)] = 2$, but because of the above argument we can replace G with $\overline{G}$ and therefore assume that $\phi(G) = A$ is abelian and of finite index m. Let $x_1 = 1$, and $x_2, ..., x_m$ be coset representatives of A. Then FG can be regarded as a free left $FA-$*module*. Given $\alpha, \beta \in FG$ with $\beta = \sum \beta_i x_i$, $\beta_i \in FA$ we have $\beta\alpha = \sum \beta_i x_i \alpha = \sum \beta_i (x_i \alpha)$. Thus right multiplication by α is an FA-homomorphism $r_\alpha : FG \to FG$, and thus r_α can be represented as an $m \times m$ FA-matrix. Thus we can define a homomorphism $\rho(\alpha) = [\alpha_{ij}] \in M_m(FA)$ so that $r_\alpha(x) = [\alpha_{ij}]x$, for all $x \in FG$. Since $r_\alpha(1) = [\alpha_{ij}](1) = \alpha$ we see that ρ is a monomorphism.

Now for each i we define a subgroup $(A, x_i) = \{a^{-1}x_i^{-1}ax_i | a \in A\}$, of G. Let $S_i = \Delta(A, x_i)$ be the augmentation ideal of $F(A, x_i)$, and then let $S = S_2...S_m$. Now note that the map $a \to (a, x_i)$ is a homomorphism from FA onto S_i because $ab \to b^{-1}a^{-1}x_i^{-1}abx_i = a^{-1}b^{-1}x_i^{-1}ax_ix_i^{-1}bx_i = a^{-1}x_i^{-1}ax_ib^{-1}x_i^{-1}bx_i$ because A is a normal abelian subgroup of G. Noticing that the kernel of this map is $\{a | a^{-1}x_i^{-1}ax_i = 1\} = \mathcal{C}_A(x_i)$ we see that $(A, x_i) \simeq A/\mathcal{C}_A(x_i)$. Thus S_i is commutative and hence S is commutative. By construction $[G : \mathcal{C}_G(x_i)]$ is infinite, and thus $[A : \mathcal{C}_A(x_i)]$ is infinite because $[G : A][A : \mathcal{C}_A(x_i)] = [G : \mathcal{C}_G(x_i)]$ and $[G : A]$ is finite. Hence we conclude that (A, x_i) is infinite, and the annihilator $A_{FA}(S_i)$ of S_i is 0. For if $\alpha S_i = 0$ then $\alpha(1 - h) = 0$ for all $h \in (A, x_i)$. Thus $\alpha = \alpha h$ and therefore for $g \in supp(\alpha)$ we have $\alpha_{gh} = \alpha_g$, which implies $gh \in supp(\alpha)$ thus $supp(\alpha)$ is infinite, this is a contradiction. Now because the annihilator $A_{FA}(S_i) = 0$ for all i we conclude that S is not nilpotent.

Thus we have shown that S is a commutative F algebra which is not

nilpotent. Let $R = (KA)\rho(KG)$ Now for $\alpha \in KA$ we have $r_\alpha(x_i^{-1}) = \alpha^{x_i}x_i^{-1}$ thus $\rho(\alpha) = diag(\alpha^{x_1}, ..., \alpha^{x_n})$. Now for any $a \in A$ we have $a^{x_i} \in A$ because A is normal hence we can view $a^{-1}a^{x_i} = a^{-1}a^{x_i} * I_n \in R$. Thus $a^{-1}(a^{x_i} - \rho(a)) = diag(a^{-1}(a^{x_i} - a^{x_1}, ..., a^{-1}(a^{x_i} - a^{x_n})$. Therefore letting $i \geq 2$ we have zero in the i position and $a^{-1}(a^{x_i} - a) = (a, x_i) - 1$ in the first position. Thus given any $s_i \in S_i$ we can define $\alpha_i \in R$ which is a diagonal matrix with s_i in the first position and zero in the i position. Now let $\alpha = \alpha_2...\alpha_n$, then $\alpha \in R$ and is of the form $\alpha = diag(s, 0, ..., 0)$ for any $s \in S$. Let e_{ij} be the standard matrix units then $\rho(x_i)e_{11} = e_{i1}$ the first column of $\rho(x_i)$ is precisely the first column of e_{i1}, similarly $e_{11}\rho(x_i^{-1}) = e_{1i}$, because the first row of $\rho(x_i^{-1})$ is the same as the first row of e_{1i}. Thus $R \supseteq \rho(x_i S e_{11}\rho(x_j^{-1}) = S\rho(x_i)e_{11}e_{11}\rho(x_j^{-1}) = Se_{ij}$, because $e_{11}e_{11} = e_{11}$. Therefore $R \supseteq M_n(S)$.

Now R satisfies the same identity as FG because FA is central, and thus so does $M_m(S)$. Thus applying the previous lemma yields the result.

Now if $p = 0$ then $\mathbb{Z} \subseteq F$, and thus the identities hold for $(\mathbb{Z}/3\mathbb{Z})G$ and the above arguments yield the result. □

Remark 1.3. ([16] lemma 1.4.7.) A finitely generated FC-group G is residually finite.

Definition 1.9. Given a group G and some prime p, we say that G is p-abelian if G' is a finite p-group.

For convenience of notation we take 0-abelian to mean abelian.

Lemma 1.8. *Let $char(F) = p$. If FG is Lie nilpotent or Lie solvable and $p \neq 2$ then G is p-abelian.*

Proof. Our goal is to show that G' is a finite p-group, but since FG satisfies a polynomial identity we know that $|\phi(G)'| < \infty$ and G is an FC group by lemma 1.7 thus $G = \phi(G)$, therefore $|G'| < \infty$.

Assume that $p \geq 2$. For any $g = a^{-1}b^{-1}ab \in G'$ we let $H = \langle a, b \rangle$. Then H is also an FC group and since it is finitely generated it is residually finite by remark 1.3, and we can assume that it is finite. We can also assume that F is

algebraically closed because otherwise if $F' \supset F$ and F' is algebraically closed then $F'G$ is Lie nilpotent or Lie solvable. Given any irreducible representation ψ of FH we have $\psi(FG)$ is a finite dimensional semisimple algebra over F and thus $\psi(FG) \simeq M_m(F)$ and thus $m = 1$ by lemma 1.6 because $\psi(FG)$ is Lie nilpotent or Lie solvable and thus so is $M_m(F)$.

Since $m = 1$ we have $\psi(a^{-1}b^{-1}ab) = \psi(a^{-1})\psi(b^{-1})\psi(a)\psi(b) = 1$ because F is commutative. Thus $\psi(h-1) = 0$ for all $h \in H'$, and for all irreducible representations ψ of FH and hence $h-1 \in J(FH)$. This implies that $\Delta(H') \subseteq J(FH)$ and hence $\Delta(H')$ is nilpotent. This implies that H' is a p-group, by lemma 1.4.

If $char(F) = 0$ then we have $\mathbb{Z}G$ satisfies the same polynomial identities that FG does and hence so does $(\mathbb{Z}G/p\mathbb{Z})G$ and thus G' is a p-group for all primes p, ie., G' is trivial, thus G is abelian. □

Lemma 1.9. *If G is p-abelian, then FG is Lie solvable for any field F with $char(F)=p$.*

Proof. Considering the standard map $\varphi : FG \to F(G/G')$ we notice that if $a, b \in FG$ then $\varphi(ab - ba) = \varphi(a)\varphi(b) - \varphi(b)\varphi(a) = 0$ because $F(G/G')$ is commutative. But then $[FG, FG] \subseteq ker(\varphi) = \Delta(G, G') = \Delta(G')FG$. Thus we have $\delta^n(FG) \subseteq (\Delta(G'))^{2^n}FG$ because if we inductively assume this result for $n-1$ then for $x \in [(\Delta(G'))^{2^{n-1}}FG, (\Delta(G'))^{2^{n-1}}FG]$, we have $x = a'ab'b - b'ba'a$ for some $a', b' \in (\Delta(G'))^{2^{n-1}}$. This implies that $a'a, b'b \in (\Delta(G'))^{2^{n-1}}FG$. Thus $a'ab'b \in ((\Delta(G'))^{2^{n-1}}FG)^2 = (\Delta(G'))^{2^n}FG$ because this is an ideal and hence $x \in (\Delta(G'))^{2^n}FG$. Now by assumption we have that G' is p-abelian, and hence $\Delta(G')$ is nilpotent by lemma 1.4. Thus $\delta^n(FG) \subseteq (\Delta(G'))^{2^n}FG = 0$ for some n. □

Remark 1.4. ([16] lemma 5.4.2.) Any p-group with a finite commutator subgroup is nilpotent.

Lemma 1.10. *If a group algebra FG is Lie nilpotent then G is nilpotent.*

Proof. If $char(F) = 0$ then G is abelian by lemma 1.8, and hence nilpotent. Therefore we assume $char(F) = p \geq 2$ then choosing m so that p^m is greater than the index of Lie nilpotence of FG, we have $[x,_{p^m} y] = [x, y^{p^m}] = 0$ for any $x, y \in G$, thus y^{p^m} is central. Therefore if we let $\overline{G} = G/Z(G)$ then $\overline{G}$ is a p-group because $\overline{y}^{p^m} = \overline{y^{p^m}} = \overline{1}$. Also from lemma 1.7 we know that $G = \phi(G)$ and thus G' is finite because FG satisfies a polynomial identity. Hence $\overline{G}'$ is also finite. But any p-group with a finite commutator subgroup is nilpotent by remark 1.4, therefore $\overline{G}$ is nilpotent, and this implies that G is nilpotent. □

Lemma 1.11. *If G is nilpotent and p-abelian then FG is Lie nilpotent for any field F with char(F)=p.*

Proof. Suppose F and G satisfy the hypothesis. We will proceed by induction on the order of G'. The result is clear if $|G'| = 1$ because G is abelian. Now $G' \bigcap Z(G) \neq 1$ because G is nilpotent. So suppose $|G'| = p^m$ let $z \in G' \bigcap Z(G)$ with $o(z) = p$. Then $\overline{G} = G/\langle z \rangle$ satisfies the hypothesis, and $|\overline{G}'| < |G'|$ thus $F\overline{G}$ is Lie nilpotent. Suppose the index of Lie nilpotence is n. Then $[K\overline{G},_n K\overline{G}] = 0$, ie. $[KG,_n KG] \subseteq \Delta(G, \langle z \rangle) = (1 - z)KG$. Now since z is central with order p, we have $(1 - z)^p = 0$. Thus $[KG,_{np} KG] \subseteq (1-z)^p KG = 0$. □

The following result is needed in characterizing Lie solvable group algebras over fields of characteristic 2. This is the first part of the proof [13] lemma 3.4.

Remark 1.5. Suppose $char(F) = 2$ and FG is Lie solvable. If the maximal normal 2-subgroup of G is trivial then G has an abelian subgroup of index at most 2.

Lemma 1.12. *Let char(F)=2. FG is Lie solvable if and only if G has a 2-abelian subgroup of index 1 or 2.*

Proof. ($\Leftarrow$) If G is 2-abelian then FG is Lie solvable by lemma 1.9. Thus assume H is a subgroup which is 2-abelian, and $[G : H] = 2$. Then $\Delta(G, H') = \Delta(H')FG$ is nilpotent because H' is a finite 2-group by lemma 1.4. Suppose H is abelian then we can define a monomorphism $\rho : FG \to M_2(FH)$ as in lemma

1.7. Now $M_2(FH)$ is Lie solvable because with notation as in lemma 1.6 we have $[M_2(FH), M_2(FH)]$ lies in the span of $(FH)^2e_{12}, (FH)^2e_{21}, (FH)^2(e_{11} + e_{22})$. Noting that $e_{11} + e_{22}$ is the identity matrix and therefore central we notice that $\delta^2(M_2(FH) \subseteq [(FH)^2e_{12}, (FH)^2e_{21}] = (FH)^4(e_{11} + e_{22})$. Hence $\delta^3(M_2(FH)) = 0$. This implies that FG is Lie solvable. If H is not abelian then we have that $F(G/H')$ is Lie solvable by the previous arguments and $\Delta(G, H') = \Delta(H')FG$ is nilpotent by lemma 1.4. Thus $\delta^n(F(G/H')) = 0$ ie. $\delta^n(FG/H') \subseteq \Delta(G, H')$. Now $\delta^k(\Delta(G, H')) \subseteq (\Delta(G, H'))^{2^k} = 0$ for some k. Thus $\delta^{nk}(FG) \subseteq \Delta(G, H')^{2^n} = 0$.

($\Rightarrow$) Now suppose $char(F) = 2$ and FG is Lie solvable. Let W be the maximal normal 2-subgroup of G. If $W = \langle 1 \rangle$ then G has an abelian subgroup of index 2 by remark 1.5. Let $\overline{G} = G/W$ then $\phi(\overline{G}) = \phi(G)/W$ and $\phi(\overline{G})' = \phi(G)'/W$ thus the maximal normal 2-subgroup of $\overline{G}$ is trivial, and hence there exists an abelian subgroup $\overline{A}$ of $\overline{G}$ of index 2. Let $A = \{a \in G | \overline{a} \in \overline{A}\}$, then $\overline{A'}$ is trivial thus A' is a subset of W and hence A is 2-abelian. Now suppose $x, y \notin A$ then $\overline{x}\overline{A} = \overline{y}\overline{A}$ thus $\overline{xy^{-1}A} = \overline{A}$ hence $\overline{xy^{-1}} \in \overline{A}$ and this implies that $xy^{-1} \in A$ thus x and y are in the same coset of A, hence $[G : A] \leq 2$. Therefore G has a 2-abelian subgroup of index 2. □

The following theorems simply collect the main results of the lemmas in this section to give necessary and sufficient conditions for the Lie nilpotence or Lie solvability of FG.

Theorem 1.4. *A group algebra FG is Lie nilpotent if and only if G is p-abelian and nilpotent where $p = char(F)$.*

Theorem 1.5. *A group algebra FG is Lie solvable if and only if G is is p-abelian if $char(F) = p \neq 2$, or G has a 2-abelian subgroup of index at most 2 if char(F)=2.*

Chapter 2

Lie Nilpotency and Lie Solvability of FG

2.1 G contains no 2-elements

Every group algebra has a standard involution based on the inverses of the group elements. This involution naturally gives rise to the study of two subsets of the group algebra, namely the symmetric and skew symmetric elements. In many cases if one of these sets is Lie solvable or Lie nilpotent, the whole group algebra is actually Lie solvable or Lie Nilpotent. The results about Lie nilpotence in this section can be found in [7].

Definition 2.1. The standard involution on FG is given by $(\sum \alpha_g g)* = \sum \alpha_g g^{-1}$

Definition 2.2. A group algebra FG is said to satisfy a *-polynomial identity if there exists a non-zero $f(x_1, x_1^*, ..., x_n, x_n^*) \in K\langle x_1, x_1^*, ..., x_n, x_n^* \rangle$ satisfying $f(\alpha_1, \alpha_1^*, ..., \alpha_n, \alpha_n^*) = 0$ for all $\alpha_1, \alpha_1^*, ..., \alpha_n, \alpha_n^* \in KG$

Remark 2.1. It is useful to note that if a group algebra FG satisfies a *-polynomial identity, then it also satisfies a regular polynomial identity. [2] Theorem 1.

Definition 2.3. The set $FG^+ = \{x \in FG | x^* = x\}$ is called the set of symmetric elements of FG and the set $FG^- = \{x \in FG | x^* = -x\}$ is called the set of skew symmetric elements.

Let us denote the center of G by $Z(G)$.

Lemma 2.1. *Let $H = \{g^2 | g \in Z(G)\}$ and assume $|H| = \infty$. If $\alpha \in FG$ with $(z^2 - 1)\alpha = 0$ for all $z \in Z(G)$ then $\alpha = 0$.*

Proof. First note that for $g, h \in Z(G)$ we have $g^2h^2 = (gh)^2$ therefore H is a group.

Assume $\alpha \in FH$, then $\alpha g = \alpha$ all $g \in H$. But then if $h \in supp(\alpha)$ so is hg for all $g \in H$, and thus the $|supp(\alpha)| = \infty$ because H is infinite, this is a contradiction and therefore $\alpha = 0$.

For $\alpha \in FG$ we write $\alpha = \sum \alpha_i x_i$ with $\alpha_i \in FH$ and $x_i \in \mathcal{T}$ a right transversal of H. Clearly if $x_i = 1$ for some i, then $(z^2-1)\alpha = 0 \Rightarrow (z^2-1)\alpha_i = 0$. Hence by looking at $x_i^{-1}\alpha$ we see that $\alpha_i = 0$, and we can do this for any i, therefore $\alpha = 0$. □

Lemma 2.2. *Let $f = \sum \alpha_\sigma x_{\sigma(1)}^{a_{1\sigma}} x_{\sigma(2)}^{a_{2\sigma}} ... x_{\sigma(n)}^{a_{n\sigma}}$ with $\sigma \in S_n$, and $a_{i\sigma} \in \{1, *\}$ and let g = sum of all monomials in f which do not contain x_i^* for any i. If $g \neq 0$ and f is a *-polynomial identity for FG, then so is $(z_1^2 - 1)...(z_n^2 - 1)g$ for any $z_i \in Z(G)$.*

Proof. Fix i and let $f = f_1 + f_2$ where f_1 is the sum of all monomials in which x_i appears and f_2 is the sum of all monomials in which x_i^* appears. Now $f_1/neq0$ because $g \neq 0$. Let $z \in Z(G)$, then $f(x_1, ..., zx_i, ..., x_n) = zf_1 + z^*f_2$ is a *-polynomial identity for FG, and clearly $z^*f = z^*f_1 + z^*f_2$ is a *-polynomial identity for FG. Hence subtracting these 2 we get that $(z - z^*)f_1$ vanishes on FG. Applying this process for each i and noting that $z^* = z^{-1}$ shows that $(z_1 - z_1^{-1})...(z_n - z_n^{-1})g$ vanishes on FG, and since each z_i is central multiplying through by $z_1 z_2 ... z_n$ gives $(z_1^2 - 1)...(z_n^2 - 1)g = 0$ for any $z_i \in Z(G)$. □

Corollary 2.1. *Assume $|Z(G)^2| = \infty$. Let $f = \sum \alpha_\sigma x_{\sigma(1)}^{a_{1\sigma}} x_{\sigma(2)}^{a_{2\sigma}} ... x_{\sigma(n)}^{a_{n\sigma}}$ with $\sigma \in S_n$, and $a_{i\sigma} \in \{1, *\}$ and let g = sum of all monomials in f which do not*

*contain x_i^* for any i. If $g \neq 0$ and f is a *-polynomial identity for FG, then g is a polynomial identity for FG.*

Proof. From previous lemma $(z_1^2 - 1)...(z_n^2 - 1)g$ vanishes on FG for all $z_i \in Z(G)$. Thus inductively we can assume $(z_i^2 - 1)...(z_n^2 - 1)g$ vanishes on FG for some i and then application of lemma 2.1 implies that $(z_{i+1}^2 - 1)...(z_n^2 - 1)g$ vanishes on FG therefore after n steps we arrive at g vanishes on FG. □

Corollary 2.2. *Assume $|Z(G)^2| = \infty$. If FG^+ or FG^- is Lie nilpotent of index n, then FG is Lie nilpotent of index n.*

Proof. $[x_1 + x_1^*, ..., x_n + x_n^*]$ or $[x_1 - x_1^*, ..., x_n - x_n^*]$ is a *-polynomial identity for FG, and application of corollary 2.1 proves the result. □

Corollary 2.3. *Assume $|Z(G)^2| = \infty$. If FG^+ or FG^- is Lie solvable of index n, then FG is Lie solvable of index n.*

Proof. $[...[x_1 + x_1^*, x_2 + x_2^*], ..., [x_{n-1} + x_{n-1}^*, x_n + x_n^*]...]$ or $[...[x_1 - x_1^*, x_2 - x_2^*], ..., [x_{n-1} - x_{n-1}^*, x_n - x_n^*]...]$ is a *-polynomial identity for FG, and application of corollary 2.1 again proves the result. □

Lemma 2.3. *Let F be a field with $char(F) \neq 2$ such that $M_n(F)$ has an involution which is either transpose type or symplectic.*

(i) If $M_n(F)^-$ is Lie solvable or Lie nilpotent then $[M_n(F)^-, M_n(F)^-] = 0$

(ii) If $M_n(F)^+$ is Lie nilpotent then $[M_n(F)^+, M_n(F)^+] = 0$

Proof. First notice that if a set is Lie nilpotent of index n then $[...[[x_1, x_2], [x_3, x_4]], ..., [[x_{2^n-3}, x_{2^n-2}], [x_{2^n-1}, x_{2^n}]]...] = 0$, and thus the set is also Lie solvable. Therefore assume that $M_n(F)^+$ or $M_n(F)^-$ is Lie solvable.

Now suppose the involution on $M_n(F)$ is a transpose type involution, defined by, $x^* = A\overline{x}A^{-1}$ where A is a diagonal matrix and $\overline{x} = [x_{ji}^*]$ with $x = [x_{ij}]$. We will write A_i for the entry in the ii position of the matrix A, and e_{ij} for the matrix with 1 in the ij position and 0 elsewhere. With the definition of $*$ we can check to see that $e_{ij}^* = A_j A_i^* e_{ji}$ because $1^* = 1$ in any field. We now define $z_{ij} = e_{ij} + e_{ij}^*$ and $y_{ij} = e_{ij} - e_{ij}^*$, and note that these are symmetric or

skew symmetric, respectively. If $n \geq 3$ then we choose distinct i, j and k and calculate $z_{ij}z_{ki} = e_{kj} = y_{ij}y_{ki}$ and $z_{ij}z_{jk} = A_kA_i^{-1}e_{ki} = y_{ij}y_{jk}$. Thus we have $[z_{ij}, z_{ki}] = y_{kj} = [y_{ij}, y_{ki}]$. This implies that $\delta^1(M_n(F)^+, M_n(F)^+) \supseteq \{y_{ij} | 1 \leq i, j \leq n, i \neq j\}$ and that $\delta^1(M_n(D)^-, M_n(D)^-) \supseteq \{y_{ij} | 1 \leq i, j \leq n, i \neq j\}$. Thus inductively assume this holds for $\delta^{r-1}(M_n(F)^{\pm})$, then $y_{kj} = [y_{ij}, y_{ki}]$ implies that $\delta^r(M_n(F)^+, ..., M_n(F)^+) \supseteq \{y_{ij} | 1 \leq i, j \leq n, i \neq j\} \neq 0$ and $\delta^r(M_n(F)^-, ..., M_n(F)^-) \supseteq \{y_{ij} | 1 \leq i, j \leq n, i \neq j\} \neq 0$. This is a contradiction as one of these sets is assumed to be Lie solvable. Therefore if $M_n(F)$ has a transpose involution $n \leq 2$. If $n = 1$ there is nothing to do. Therefore we assume that $n = 2$ and let $a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ c_1^{-1}c_2 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 1 & (1/2)c_1c_2^{-1} \\ 1/2 & 2 \end{pmatrix}$, and $d = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ then a, b, c, d are symmetric. Now $[a, b] = \begin{pmatrix} 0 & 1 \\ -c_1^{-1}c_2 & 0 \end{pmatrix}$ and $[a, b, c] = \begin{pmatrix} 1 & 1 \\ c_1^{-1}c_2 & -1 \end{pmatrix}$, and $[a, b, c, d] = \begin{pmatrix} 0 & 1 \\ -c_1^{-1}c_2 & 0 \end{pmatrix} = [a, b]$. This implies inductively that $[a, b, c, d, c, d...] \neq 0$ and therefore the symmetric elements are not Lie nilpotent, therefore if the symmetric elements are assumed to be Lie nilpotent we conclude that $n = 1$ and everything commutes. Let $J = \begin{pmatrix} 0 & 1 \\ -c_1^{-1}c_2 & 0 \end{pmatrix}$ and notice that with this involution generic skew symmetric elements are of the form $x = \begin{pmatrix} 0 & a \\ -c_1^{-1}c_2a & 0 \end{pmatrix} = aJ$, thus $[aJ, bJ] = ab[J, J] = ab(0) = 0$ therefore the skew symmetric elements commute.

Otherwise $M_n(F)$ has a symplectic involution and n is even, suppose $n \geq 4$. The symplectic involution is defined by $[A_{ij}]^* = [A_{ji}^*]$ where A_{ij} are the 2×2 sub-matrices and A_{ij}^* is defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Again we will assume Lie solvability of either the symmetric or skew symmetric elements. Let $a = e_{13} + e_{42}$, $b = e_{24} + e_{31}$, and $I_2 = e_{11} + e_{22}$, and notice that these

are all symmetric elements. Let $a^- = e_{13} - e_{42}$, $b^- = e_{24} - e_{31}$, and $I_4^- = -e_{11}+e_{22}+e_{33}-e_{44}$ and notice that these are all skew elements. Also note that since $char(F) \neq 2$ we have $1/2(I_n) \in M_n(F)$ is central and symmetric, thus if $x \in M_n(F)$ is symmetric or skew symmetric so is $(1/2)x$ because $((1/2)x)^* = (1/2)^* x^* = (1/2)x^*$.

Calculating we obtain: $[a, b] = I_4^-$, $[a, I_2] = a^-$, and $[b, I_2] = b^-$, and $[b^-, a^-] = I_4^-$, $[I_4^-, (a^-/2)] = a^-$, $[(b^-/2), I_4^-] = b^-$. Also since $1/2^k$ is central and symmetric we can multiply it by any of the terms in the Lie brackets, without changing whether it is skew or symmetric, and obtain a factor of $1/2^k$ on the right hand side of the equation.

Let $S = \{(1/2^k)I_4^-, (1/2^k)a^-, (1/2^k)b^- | k \geq 0\}$, then we now have that $[M_n(D)^+, M_n(D)^+] \supseteq S$ and similarly $[M_n(D)^-, M_n(D)^-] \supseteq S$. Then inductively if we assume $\delta^{r-1}(M_n(D)^\pm, ..., M_n(D)^\pm) \supseteq S$ the equations above imply $\delta^r(M_n(D)^\pm, ..., M_n(D)^\pm) \supseteq S \neq 0$. This is a contradiction, therefore $n \leq 2$. If $n = 1$ everything commutes, hence assume $n = 2$, and suppose $[a_{ij}]$ is symmetric, then $a_{11} = a_{22}$ and $a_{12} = -a_{12}$, and $a_{21} = -a_{21}$ this implies that $a_{12} = a_{21} = 0$ thus $[a_{ij}] = a_{11} * I$ and therefore the symmetric elements commute. Now let us consider the skew symmetric case. The elements $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are all skew symmetric, and $[b, a] = 2c$, $[c, a] = 2b$, and $[b, c] = 2a$. As above this implies that $[M_2(D)^-, M_2(D)^-] \supseteq S = \{(1/2^k)a, (1/2^k)b, (1/2^k)c | k \geq 0\}$. Thus inductively we obtain $\delta^r(M_2(D)^-, ..., M_2(D)^-) \supseteq S \neq 0$ This contradiction implies that $n = 1$ if KG^- is Lie solvable. □

Remark 2.2. If R is primitive with involution $*$ and has a minimal right ideal then, either R is isomorphic to $M_n(D)$ where D is a division ring or, for every $n > 0$, R has a subring $A(n)$, invariant under $*$, such that:

1. $A(n) \simeq M_n(D)$, if $e^* = e$ for a primitive idempotent e in R, with $*$ of transpose type, or

2. $A(n) \simeq M_{2n}(D)$, D a field, if $e^* \neq e$ for every primitive idempotent e in R, with $*$ symplectic [11] Corollary to theorem 1.2.2

Theorem 2.1. *Assume FG is semiprime with $char(F) \neq 2$, and G contains no 2-elements.*

(i) If FG^+ is Lie nilpotent then $[FG^+, FG^+] = 0$.

(ii) If FG^- is Lie nilpotent or Lie solvable then $[FG^-, FG^-] = 0$.

Proof. For any prime ideal P of FG either $P^* \subseteq P$ or $P^* \not\subseteq P$. First suppose $P^* \not\subseteq P$, then $(P + P^*)/P$ is an ideal of FG/P, and for $x \in P^*$ we have $x + x^* + P = x + P$. Now since Lie nilpotence implies Lie solvability we will assume that FG^+ is Lie solvable. Then $[...[x_1+x_1^*, x_2+x_2^*], ..., [x_{n-1}+x_{n-1}^*, x_n+x_n^*]...] = 0$. Thus $P = [...[x_1 + x_1^*, x_2 + x_2^*], ..., [x_{n-1} + x_{n-1}^*, x_n + x_n^*]...] + P = [...[x_1 + x_1^* + P, x_2 + x_2^* + P], ..., [x_{n-1} + x_{n-1}^* + P, x_n + x_n^* + P] = [...[x_1 + P, x_2 + P], ..., [x_{n-1} + P, x_n + P]$ for $x \in P^*$. Thus $(P + P^*)/P$ is Lie solvable. The same result follows if FG^- is Lie solvable because $x - x^* + P = x + P$ for $x \in P^*$. But since $(P + P^*)/P$ is a prime algebra we can conclude that it is commutative by theorem 1.3. Now for $a, b \in FG/P$ and $x, y \in (P+P^*)/P$ we have $abxy = aybx = bxay = bayx = baxy$ because $bx, ay, x, y \in (P + P^*)/P$ which is a commutative ideal. Therefore $abxy = baxy$ thus $(ab - ba)(I^2) = 0$ but since $I \neq 0$, $I^2 \neq 0$ thus $ab - ba = 0$ and FG/P is also commutative.

If $P^* \subseteq P$, then FG/P is a prime ring with induced involution, and $(FG/P)^+ = FG^+(FG/P)$ is Lie nilpotent or $(FG/P)^- = FG^-(FG/P)$ is Lie solvable. This implies FG/P satisfies a *-polynomial identity, and hence a polynomial identity by remark 2.1 and thus FG/P has a ring of quotients which is isomorphic to $M_n(D)$ where D is a division algebra which is finite dimensional over its center division algebra by remark 1.1. Suppose now that $M_n(D)$ has a primitive idempotent $e = e^* = e^2$ with $eM_n(D)$ a minimal right ideal. Then $D = eM_n(D)e$ and $D^* = e^*M_n(D)e^* = eM_n(D)e$, thus D is invariant under $*$. Then taking a maximal subfield of K of D we have $M_n(D) \otimes_{Z(D)} K \simeq M_n(D \otimes_{Z(D)} K) \simeq M_n(M_l(K)) \simeq M_r(K)$ by remark 1.2. Since D is stable under $*$ then $M_r(K)$ will also have $M_r(K)^+$ Lie nilpotent or $M_r(K)^-$ Lie solvable, and if $M_r(K)^+$ or $M_r(K)^+$ is commutative then $M_n(D)^+$ or $M_n(D)^-$ will also be commutative. Otherwise if every primitive idempotent of $M^n(D)$ has $e \neq e^*$ then D is a field by remark 2.2. Thus in any case we may

assume that D is a field. Remark 2.2 also tells us that $*$ is either transpose type or symplectic.

Hence application of the previous lemma implies $[(FG/P)^+, (FG/P)^+] = 0$ or $[(FG/P)^-, (FG/P)^-] = 0$.

So $[FG^+, FG^+] \subset P$ or $[FG^-, FG^-] \subset P$. But this holds for all P, thus $[FG^+, FG^+] \subseteq \cap P$ or $[FG^-, FG^-] \subseteq \cap P$.

Suppose that $0 \neq x \in \cap P$. We will now define a sequence $T = \{x_1, ...x_i, ..\}$ by letting $x = x_1$ and inductively let $x_i = x_{i-1}y_{i-1}x_{i-1}$ with y_{i-1} chosen so that $x_i \neq 0$. This can be done because $Rx_{i-1}R$ is a non-zero ideal, and thus it is not nilpotent, therefore it has square which is not zero. Now T is disjoint from the ideal 0 by construction, thus by Zorn's lemma there exists an ideal I of R which is maximal subject to the property that $T \cap I = \emptyset$. Let A and B be ideals of R with $A, B \not\subseteq I$. Then the ideals $A + I$ and $B + I$ properly contain I and thus have non-trivial intersection with T. Thus $x_i \in A + I$, $x_j \in B + I$, but then by definition of x_i we have that $x_n \in A + I$ and $x_n \in B + I$ for any $n \geq i, j$. Thus $x_{n+1} = x_n y_n x_n \in (A + I)(B + I) \subseteq AB + I$. This implies that $AB \not\subseteq I$ because $x_{n+1} \notin I$, thus I is prime. This is a contradiction to definition of x, thus $\cap P = 0$. This completes the proof. □

Remark 2.3. The Lie nilpotence portion of previous theorem can be generalized to any semiprime ring R with involution, satisfying $2R = R$ as in [7] theorem 1.

Lemma 2.4. *Let $H = \langle g, h \rangle$ and $[g + ag^{-1}, h + ah^{-1}] = 0$, where a is either 1 or -1 , and H has no 2-elements. Suppose FH^+ or FH^- is Lie nilpotent for some field F with $char(F) \neq 2$, then H is abelian.*

Proof. Suppose $a = 1$ then

$gh + gh^{-1} + g^{-1}h + g^{-1}h^{-1} - hg - hg^{-1} - h^{-1}g - h^{-1}g^{-1} = 0$. Therefore $gh = hg, hg^{-1}, h^{-1}g$ or $h^{-1}g^{-1}$ or 2 of the added terms (if $char(F) = 3$) for otherwise we would have $gh \in supp(0)$. In the latter case we would have that g or h is an element of order 2 as $gh = g^{-1}h$ or $gh = gh^{-1}$. If $gh = h^{-1}g^{-1}$, then gh is an element of order 2. If $gh = hg^{-1}$ then $hg = g^{-1}h$ and thus

$gh^2 = hg^{-1}h = h^2g$ and $h^2 \in Z(H)$. This implies that h has infinite order for otherwise $2|o(h)$ and $h^{o(h)/2}$ is an element of order 2. Therefore FH is Lie nilpotent by corollary 2.1, and thus H is nilpotent by theorem 1.4. But $gh = hg^{-1}$ implies $g^kh = hg^{-k}$ for any k. If $g^k \in Z(H)$ then the previous equality implies $g^{2k} = 1$ and thus $g^k = 1$ because there are no 2-elements in H. Thus $\langle g \rangle \bigcap Z(H) = 1$. Since H is nilpotent we can assume that $\langle g \rangle \bigcap Z_i(H)$ is non-trivial for some minimal $i \geq 2$, where Z_i is the i^{th} group in the upper central series. This implies $g^k \in Z_i(H)$ for some k, thus $g^kh = hg^kz$ for some $z \in Z_{i-1}(G)$. Now $g^kh = hg^{-k}$ implies $z = g^{-2k} \in Z_{i-1}(H)$ contrary to minimality of i. Thus $gh \neq hg^{-1}$. If $gh = h^{-1}g$ then $g^2 \in Z(H)$ and again H is nilpotent. So we must conclude that $gh = hg$, thus H is abelian.

Now if $a = -1$ then $gh - gh^{-1} - g^{-1}h + g^{-1}h^{-1} - hg + hg^{-1} + h^{-1}g - h^{-1}g^{-1} = 0$. Then $gh = hg, gh^{-1}, g^{-1}h$ or $h^{-1}g^{-1}$ or two of the added terms when $char(F) = 3$. If gh equals two of the added then $gh = hg^{-1}$ or $gh = h^{-1}g$ but we have seen this is impossible in the $a = 1$ case. Similarly $gh \neq h^{-1}g^{-1}$ from the arguments in the $a = 1$ case. Now if $gh = gh^{-1}$ then $h^2 = 1$ and if $gh = g^{-1}h$ then $g^2 = 1$ both of these are impossible as H contains no 2-elements. Thus again $gh = hg$ and H is abelian. □

Theorem 2.2. *Assume $char(F) = 0$ and G has no 2-elements. If FG^+ is Lie nilpotent or FG^- is Lie solvable then FG is commutative and therefore Lie nilpotent and Lie solvable.*

Proof. $Char(F) = 0$ implies FG is semiprime by theorem 1.2. Therefore theorem 2.1 yields $[g + ag^{-1}, h + ah^{-1}] = 0$ for $a = 1$ or -1. Let $H = \langle g, h \rangle$ for some $g, h \in G$. Then lemma 2.4 shows that $gh = hg$ hence G is abelian. This implies FG is commutative. □

Remark 2.4. Let $R = KG$ be the group algebra of a group G over a field K and A an additive subgroup of R. Then $[A,_n R] = 0$ for some $n \Rightarrow [A, R]R$ is an (associative) nilpotent ideal of R. This is the main theorem in [6].

Lemma 2.5. *Suppose G is a group with no 2-elements and that $char(F) =$*

$p > 2$ and FG^+ or FG^- is Lie nilpotent. If $Z(G)$ has an element z with $o(z) = q \neq p$ with q prime then G' is a finite p-group.

Proof. From lemma 2.2 we get that $(z^2-1)^n[FG,...,FG] = 0$. This implies $[A, FG]FG = (z^2-1)^n[FG, FG]FG$ is nilpotent, with $A = (z^2-1)^n FG$ by the preceding remark. Now $\Delta(G, G')$ is generated over F by $\{ghg^{-1}h^{-1} - 1 | g, h \in G\} = \{(gh - hg)g^{-1}h^{-1} | g, h \in G\} \subseteq [FG, FG]FG$ by proposition 1.1. Thus $(z^2-1)^n\Delta(G, G')$ is a nilpotent ideal of FG. Now $o(z^2) = q$ and thus $(1-z^2)^{p^m n}\Delta(G, G')^{p^m}=0$ for some m hence $(1-z^k)\Delta(G, G')^{p^m} = 0$, with $1 \leq k \leq q-1$. Now for $x \in G'$ we have $(1-z^k)(1-x)^{p^m} = (1-z^k)(1-x^{p^m}) = 0$ which implies $(1-x)^{p^m} = 0$. Thus $\Delta(G')^{p^m} = 0$, therefore $\Delta(G')$ is nilpotent and thus G' is a finite p-group by lemma 1.4. □

Lemma 2.6. *Assume $G^{p^m} \subseteq Z(G)$ and $char(F) = p \neq 2$. If FG^+ or FG^- is Lie nilpotent then G is nilpotent.*

Proof. Since FG satisfies a polynomial identity by remark 2.1, the FC subgroup ϕ is of finite index in G and $|\phi'| < \infty$ by theorem 1.1. Now G/ϕ is a finite p-group since for $\overline{g} \in G/\phi$ we have $\overline{g}^{p^m} = \overline{g^{p^m}} = \overline{1}$ because $g^{p^m} \in G^{p^m} \subseteq Z(G) \subseteq \phi$.

We now want to show that G is nilpotent. We can assume ϕ^{p^m}=1 since we can factor by $\phi^{p^m} \subseteq Z(G)$. Consider G/ϕ' acting as a finite p-group of automorphisms of ϕ/ϕ', so $(\phi/\phi', _{r_1} G/\phi') = 1$ for some r_1 since ϕ^{p^m}=1 by lemma 1.5. Hence $(\phi, _{r_1} G) \subseteq \phi'$. Now $(\phi')^{p^m} = 1$ so ϕ' is a finite p-group. As above G/ϕ'' acts as a finite abelian p-group of automorphisms of ϕ'/ϕ'', so $(\phi', _{r_2} G) \subseteq \phi''$. Thus $(\phi, _{r_1+r_2} G) \subseteq \phi''$. Continuing in this way we get $(\phi, _k G) = 1$. This implies that $\phi \subseteq Z_k(G)$ the k^{th} term of the upper central series. Since G/ϕ is a finite p-group it is nilpotent, say, of index n. Let B_i denote the inverse image of $Z_i(G/\phi)$ under the standard map $G \to G/\phi$, then $Z_k(G) \supseteq B_1$. Inductively assume that $Z_{k+i}(G) \supseteq B_i$, then for $g, h \in B_{i+1}$ we have $gh = hgbr$ with $b \in B_i \subseteq Z_{k+i}$ and $r \in \phi \subseteq B_i \subseteq (Z_{k+i})$. Thus we have $Z_{k+n} \supseteq B_n = G$. Therefore G is nilpotent. □

Theorem 2.3. *Assume G has no 2-elements and $char(F) \neq 2$. If FG^+ or FG^- is Lie nilpotent then FG is Lie nilpotent.*

Proof. If $char(F) = 0$ we are done by theorem 2.2. Therefore suppose $char(F) = p > 2$ and FG^+ or FG^- is Lie nilpotent of degree n. Pick any $g, h \in G$, and chose m so that $p^m > n$. Then $[g \pm g^{-1}, h^{p^m} \pm h^{-p^m}] = 0$ by lemma 1.3. Hence if we let $H = \langle g, h^{p^m} \rangle$ we can apply lemma 2.4 to find that H is abelian, thus $h^{p^m} g = g h^{p^m}$ and $h^{p^m} \in Z(G)$, ie. $G^{p^m} \subseteq Z(G)$. Thus the previous lemma gives that G is nilpotent.

We can assume $Z(G)$ is a finite p-group, because if $|Z(G)|=\infty$ then corollary 2.2 yields the result, and if $Z(G)$ is not a p-group, lemma 2.5 gives the result. Induct on the class of nilpotency of G and we get $(G/Z(G))'$ is a finite p-group. Thus G' is a finite p-group, proving the theorem. □

2.2 G contains 2-elements and FG^+ is Lie Nilpotent

In the previous section for the most part we were looking at fields F with $char(F) \neq 2$ and groups G which do not contain 2-elements. In this section we will continue to assume $char(F) \neq 2$ but we will allow G to contain 2-elements. We will characterize G when FG^+ is Lie nilpotent, and leave the case when FG^- is Lie nilpotent for chapter 3. The results in this section can be found primarily in [12].

Lemma 2.7. *If FG^+ is Lie nilpotent and $char(F) \neq 2$, then every element of order 2 in G is central.*

Proof. Assume $char(F) = p > 2$ and choose m so that p^m is greater than the index of Lie nilpotence of FG^+. Suppose $x, y \in G$ both with order 2, so both $x, y \in FG^+$. Now since p is odd and y has order 2 we have $y = y^{p^m}$ for any m thus $[x, y] = [x, y^{p^m}] = [x, {}_{p^m} y] = 0$ by lemma 1.3, thus x and y commute.

Now for $z \in G$ whose order is greater than 2 we have $[z + z^{-1}, y] = [z + z^{-1}, y^{p^m}] = [z + z^{-1}, {}_{p^m} y] = 0$ by lemma 1.3, so $zy + z^{-1}y - yz - yz^{-1} = 0$. Since

these terms are all in G and the elements of G are independent we must have that the added terms equal the subtracted terms thus $zy = yz$, or yz^{-1}. If $zy = yz$ we are done. If $zy = yz^{-1}$ then $1 = zyzy^{-1} = (zy)^2$, because $y = y^{-1}$ and zy has order 2, so y and zy commute. Thus $yzy = zy^2$ and therefore $yz = zy$ and so y commutes with z. Hence if y has order 2 in G it is central.

We now need to consider the case when $char(F) = 0$. Then $\mathbb{Z}G \subseteq FG$, Thus $\mathbb{Z}G^+$ is Lie nilpotent, and hence so is $((\mathbb{Z}/3\mathbb{Z})G)^+$. Now the characteristic 3 case applies from above, completing the proof. □

Lemma 2.8. *Let FG^+ be Lie nilpotent, and $char(F) \neq 2$. If $[a + a^{-1}, b + b^{-1}] = 0$, then $ab = ba$, ba^{-1}, or $b^{-1}a$, for $a, b \in G$.*

Proof. Expanding the Lie brackets we get $ab + ab^{-1} + a^{-1}b + a^{-1}b^{-1} - ba - ba^{-1} - b^{-1}a - b^{-1}a^{-1} = 0$

Either ab equals one of the subtracted terms or ab is equal to 2 of the added terms, and $char(F) = 3$ for else we would have $ab \in supp([a + a^{-1}, b + b^{-1}])$. In the latter case $ab = a^{-1}b$ or $ab = ab^{-1}$, which implies $a^2 = 1$ or $b^2 = 1$. This implies that a or b is central from previous lemma, so $ab = ba$. If $ab = b^{-1}a^{-1}$ then $(ab)^2 = 1$ thus ab is central from the previous lemma and thus $ab^2 = bab$ hence $ab = ba$. Thus the conclusion of the lemma is satisfied. □

Definition 2.4. We will denote the quaternion group of order 8 by Q_8, where $Q_8 = \langle x, y \rangle$ and satisfies the relations $o(x) = o(y) = 4$, $x^2 = y^2$ and $y^{-1}xy = x^{-1}$.

Lemma 2.9. *Let $G = \langle a, b \rangle$, with $b^{-1}ab = a^{-1}$, and $char(F) \neq 2$. If FG^+ is Lie nilpotent, then either $a^2 = 1$ (and G is abelian) or the $o(a) = 4$ and $o(b) = 4n$, for some odd n, and $\langle a, b^n \rangle \simeq Q_8$.*

Proof. If $a^2 = 1$ then a is central and thus G is abelian. Assume $a^2 \neq 1$. Let $char(F) = p > 2$. Now $b^{-1}ab = a^{-1}$ and inverting this we get $b^{-1}a^{-1}b = a$ and hence $b^{-1}(b^{-1}ab)b = a$ thus $ab^2 = b^2a$ thus $b^2 \in Z(G)$. Suppose that b has infinite order, then $|Z(G)^2| = \infty$ thus FG is Lie nilpotent by corollary 2.2, hence G is nilpotent and G' is a finite p-group by theorem 1.4. Since G is

nilpotent $\langle a^2 \rangle$ contains a nontrivial central element, and $a^2 = ba^{-1}b^{-1}a \in G'$. So $o(a^2) = p^m$ for some $m \geq 0$ and $a^{2i} \in Z(G)$ but then $b^{-1}a^{2i}b = a^{-2i}$ implies $a^{4i} = 1$ and hence $a^{2i} = 1$ since the order of a^{2i} is odd. This contradicts $\langle a^2 \rangle$ having a nontrivial central element, hence $o(b) < \infty$.

Now $b^2 \in Z(G)$, so, $o(b) = 2k$ for some k, hence b^k is also central because it has order 2, thus $4|o(b)$ for else b would be central. Write $o(b) = 2^r n$ with n odd, and choose m so that p^m is greater than the index of Lie nilpotence of FG^+, then $0 = [ab^n + b^{-n}a^{-1}, b^n + b^{-n}, ..., b^n + b^{-n}] = [ab^n + b^{-n}a^{-1}, b^{p^m n} + b^{-p^m n}]$. Now applying the previous lemma we have either,

$ab^{n(1+p^m)} = b^{np^m}ab^n \Rightarrow ab^{np^m} = b^{np^m}a \Rightarrow b^{np^m}$ is central thus b is central because np^m is odd, a contradiction;

or $ab^{n(1+p^m)} = b^{n(p^m-1)}a^{-1} \Rightarrow a^2 = b^{-2n}$ because $b^{n(p^m-1)}$ is central since $n(p^m - 1)$ is even;

or $ab^{n(1+p^m)} = b^{-np^m}ab^n \Rightarrow ab^{n(1+p^m)} = b^{-np^m}ab^{np^m}b^{n(1-p^m)} = a^{-1}b^{n(1-p^m)}$, hence $a^2 = b^{-2np^m}$.

Therefore, $a^2 = b^{-2n}$ or b^{-2np^m}, hence a^2 is central because b^2 is central, hence $a^{-2} = b^{-1}abb^{-1}ab = b^{-1}a^2b = a^2$ so $o(a) = 4$. Thus $b^{4np^m} = a^4 = 1$, but since $o(b) = 2^r n$ we get $o(b) = 4n$. We now have $o(a) = o(b^n) = 4$, $a^2 = b^{2n}$ and $b^{-n}ab^n = a^{-1}$, hence $\langle a, b^n \rangle \simeq Q_8$.

If $char(F) = 0$ we notice that the hypothesis is satisfied by $(\mathbb{Z}/3\mathbb{Z})G$ and thus the above arguments yield the result. □

Lemma 2.10. *Let $G = \langle a, b \rangle$, and suppose $[a + a^{-1}, b + b^{-1}] = 0$. If $Q_8 \not\subseteq G$, and $char(F) \neq 2$, then FG^+ is Lie nilpotent if and only if G is abelian.*

Proof. Either $ab = ba, b^{-1}a$ or ba^{-1} by lemma 2.8. If $ab = ba$ we have nothing to do. If $ab = b^{-1}a$ then $aba^{-1} = b^{-1}$ and if $ab = ba^{-1}$ then $b^{-1}ab = a^{-1}$, in either case lemma 2.9 yields the result, because $Q_8 \not\subseteq G$. □

Lemma 2.11. *Suppose $Q_8 \not\subseteq G$, and FG^+ is Lie nilpotent with $char(F) = p > 2$, then $G^{p^m} \subseteq Z(G)$ for some m.*

Proof. Choose m so that p^m is greater than the index of Lie nilpotence of FG^+. For $a, b \in G$ we have $0 = [a + a^{-1}, b + b^{-1}, ..., b + b^{-1}] = [a + a^{-1}, b^{p^m} + b^{-p^m}]$.

Hence with $H = \langle a, b^{p^m} \rangle$ the previous lemma yields H is abelian thus $ab^{p^m} = b^{p^m}a$. Therefore $b^{p^m} \in Z(G)$. □

Theorem 2.4. *Suppose $Q_8 \not\subseteq G$, and FG^+ is Lie nilpotent with $char(F) = p \neq 2$, then FG is Lie nilpotent.*

Proof. If $char(F) = 0$, then $\mathbb{Z}G^+$ is Lie nilpotent which implies $((\mathbb{Z}/q\mathbb{Z})G)^+$ is Lie nilpotent for any prime q so the previous lemma says $G^{q^m} \subseteq Z(G)$, since this holds for any prime q, G is abelian.

If $char(F) = p > 2$ then $G^{p^m} \subseteq Z(G)$ thus G is nilpotent by lemma 2.6. We can also assume that G is torsion for if it had an element x of infinite order, $x^{p^m} \in Z(G)$ and then $|Z(G)^2| = \infty$ and corollary 2.3 would yield the result. But if G is torsion nilpotent it can be written as a direct product $\prod_q P_q$ over all primes q where P_q is the unique Sylow q-subgroup. Since this product is direct, if g and h have orders which are powers of different primes then $g^{-1}h^{-1}gh = 1$ thus $G' = \prod_q P_q'$. Now if $q \neq p$ then $P_q = P_q^{p^m} \subseteq Z(G)$ thus $P_q' = 1$. This implies that $G' = P_p'$, and as P_p has no 2-elements theorem 2.3 tells us that FP_p is Lie nilpotent and thus P_p' is finite by theorem 1.4. This completes the proof. □

Lemma 2.12. *Let $G = Q_8 \times C$ where C is cyclic, and suppose $char(F) = p > 2$. If FG^+ is Lie nilpotent then C has order p^m or $2p^m$ for some $m > 0$.*

Proof. Write $Q_8 = \langle g, h \rangle$, and $C = \langle c \rangle$, and choose m so that p^m is greater than the index of Lie nilpotence of FG^+. Then $0 = [gc + g^{-1}c^{-1}, {}_{p^m} hc + h^{-1}c^{-1}] = [gc + g^{-1}c^{-1}, h^{p^m}c^{p^m} + h^{-p^m}c^{-p^m}]$ by lemma 1.3 and thus by lemma 2.8 we have 3 possibilities:

(*i*) if $gh^{p^m}c^{p^m+1} = h^{p^m}gc^{p^m+1}$ then $gh^{p^m} = h^{p^m}g$ but this is impossible because p^m is odd and $\langle g, h \rangle = Q_8$;

(*ii*) if $gh^{p^m}c^{p^m+1} = h^{p^m}g^{-1}c^{p^m-1}$, then $gh^{-p^m}gh^{p^m} = c^2 = 1$ so $c^2 = 1$;

(*iii*) if $gh^{p^m}c^{p^m+1} = h^{-p^m}gc^{1-p^m}$ then $g^{-1}h^{p^m}gh^{p^m} = c^{-2p^m} = 1$ hence $c^{2p^m} = 1$. □

Lemma 2.13. *Suppose $Q_8 = \langle g, h\rangle \subseteq G$, $char(F) = p > 2$, and FG^+ is Lie nilpotent. If $b \in G$ and b does not centralize $\langle g, h\rangle$, then $o(b) = 4p^m$ for some $m > 0$.*

Proof. Choose k so that p^k is greater than the index of Lie nilpotence of FG^+. Then $0 = [b + b^{-1}, {}_{p^k} g + g^{-1}] = [b + b^{-1}, g^{p^k} + g^{-p^k}] = [b + b^{-1}, g + g^{-1}]$ the first equality is by lemma 1.3 and the second because p is odd $o(g) = 4$ implies $g^{p^k} + g^{-p^k} = g^{\pm 1} + g^{\mp 1} = g + g^{-1}$. Examining the 3 cases from lemma 2.8 we find if,

(*i*) $bg = gb^{-1}$ then lemma 2.9 says $o(b) = 4$ since b is not central;

(*ii*) $bg = g^{-1}b$ then lemma 2.9 says $o(b) = 4n$ with n odd and $\langle g, b^n\rangle \simeq Q_8$. Now $bg = g^{-1}b$ implies $gb = bg^{-1}$ thus $b^2g = bg^{-1}b = gb^2$, therefore b^2 and hence b^4 centralize $\langle g, b^n\rangle$, and $o(b^4) = n$ hence $\langle b^4\rangle \bigcap \langle g, b^n\rangle = 1$. Therefore $\langle b, g\rangle = \langle g, b^n\rangle \times \langle b^4\rangle \simeq Q_8 \times C_n$ and the previous lemma gives $n = p^m$ because n is odd;

(*iii*) $bg = gb$.

The same arguments repeat if we replace g with h. Since $bg = gb$ and $bh = hb$ cannot both be true if b does not centralize $\langle g, h\rangle$ we get $o(b) = 4p^m$ with $m \geq 0$. □

Definition 2.5. A group G is called Hamiltonian if every subgroup of G is normal but G is not abelian.

Remark 2.5. Hamiltonian 2-groups are always of the form $Q_8 \times E$ where every $g \in E$ has order 2 by [9], theorem 12.5.4.

Lemma 2.14. *Suppose $Q_8 \subseteq G$, $char(F) = p > 2$, and FG^+ is Lie nilpotent, then the 2-elements form a Hamiltonian 2-group which is normal in G.*

Proof. Consider g, h 2-elements in G, and choose m so that p^m is greater than the index of Lie nilpotence of FG^+ and $p^m \equiv 1 (mod\ o(h))$. Then $0 = [g + g^{-1}, {}_{p^m} h + h^{-1}] = [g + g^{-1}, h^{p^m} + h^{-p^m}] = [g + g^{-1}, h + h^{-1}]$ by lemma 1.3, and again from lemma 2.8 we have either $gh = hg$, $gh = hg^{-1}$, or $gh = h^{-1}g$.

If $gh = hg^{-1}$ or $gh = h^{-1}g$ then lemma 2.9 gives $gh = hg$ or $\langle g, h\rangle \simeq Q_8$.

So either $gh = hg$ or $\langle g, h\rangle \simeq Q_8$. Hence $g^{-1}hg = h^{\pm 1}$ so the 2-elements form a subgroup H with every cyclic subgroup normal in H and hence every subgroup is normal in H. H is clearly not abelian as $Q_8 \subseteq G$, thus it is Hamiltonian. Normality is clear as H contains all the 2-elements and $o(g^{-1}hg) = o(h)$ for any $g, h \in G$, □

Lemma 2.15. *Suppose $Q_8 = \langle g, h\rangle \subseteq G$, $char(F) = p > 2$, and FG^+ is Lie nilpotent, then $G \simeq Q_8 \times E \times P$ where $E^2 = 1$ and P is a finite p-group.*

Proof. Suppose we have an element $x \in G$ with odd prime order $q \neq p$ or infinite order. Then x must centralize $\langle g, h\rangle$ by lemma 2.13 and $\langle x\rangle \bigcap \langle g, h\rangle = 1$, thus $Q_8 \times \langle x\rangle \subseteq G$. Then lemma 2.12 says $o(x) = p^m$ or $2p^m$, thus no such x exists. By lemma 2.14 the 2-elements form a Hamiltonian 2-group, and thus they are of the form $Q_8 \times E$ with $E^2 = 1$, thus we need only show that the p-elements in G form a finite subgroup. Consider any p-elements $x, y \in G$ and suppose $\langle x, y\rangle$ contains a 2-element. Lemma 2.13 tells us that both x and y commute with, $Q_8 = \langle g, h\rangle$ thus so would any 2-elements in $\langle x, y\rangle$. The only 2-elements in $Q_8 \times E$ that commute with $\langle g, h\rangle$ have order 2, hence any 2-elements in $\langle x, y\rangle$ must have order 2. This implies that $Q_8 \not\subseteq \langle x, y\rangle$ so we can apply theorem 2.4 to $F\langle x, y\rangle$ and therefore $F\langle x, y\rangle$ is Lie nilpotent. Hence $\langle x, y\rangle$ is nilpotent by theorem 1.4. Since $\langle x, y\rangle$ is a nilpotent group generated by p-elements, it is a p-group.

Now $G \simeq Q_8 \times E \times P$, with $E^2 = 1$ and P is a p-group. Suppose P is infinite. Then FP^+ is Lie nilpotent, and by theorem 2.3 FP is Lie nilpotent hence by theorem 1.4 P' is a finite p-group. Let $H = Q_8 \times E \times P/P'$ then FH^+ is Lie nilpotent, and so FH is Lie nilpotent by corollary 2.2 because $|Z(H)^2| = \infty$ since P/P' is an infinite abelian p-group. This is a contradiction to theorem 1.4 since H' is not a p-group. Thus P is finite. □

Theorem 2.5. *If $Q_8 \subseteq G$, and $char(F) \neq 2$ then FG^+ is Lie nilpotent if and only if $G = Q_8 \times E \times P$ where $E^2 = 1$ and P is either 1 or a finite p-group if $char(F) = p > 2$.*

Proof. Suppose G is of the form above with $|P| = p^m$. If $m = 0$ then $F(Q_8 \times E)^+ = (FE(Q_8))^+$, and the symmetric elements commute because they are generated by the elements of order 2 which are central and the elements $g+g^{-1}$ and $h + h^{-1}$ with $Q_8 = \langle g, h \rangle$. These latter commute as can be seen by $h^{-1}(g + g^{-1})h = g^{-1} + g$ and similarly $g^{-1}(h + h^{-1})g = h^{-1} + h$.

Assume $char(F) = p > 2$. We will now proceed inductively assuming $[\alpha_1, ..., \alpha_{p^{m-1}+1}] = 0$. Let $m \geq 1$ and choose $z \in Z(G)$ with $o(z) = p$. Consider $F(Q_8 \times E \times (P/\langle z \rangle))$, then $[\overline{\alpha_1}, ...\overline{\alpha_{p^{m-1}+1}}] = 0$ or $[\alpha_1, ..., \alpha_{p^{m-1}+1}] \in \Delta(G, \langle z \rangle) = (z^2 - 1)(z^{-1})(z)FG = (z - z^{-1})FG$ because $\langle z \rangle = \langle z^2 \rangle$. So $[\alpha_1, ..., \alpha_{p^{m-1}+1}] = (z - z^{-1})\omega$ for some $\omega \in FG$. If $\alpha, \beta \in FG^+$ then $(\alpha\beta - \beta\alpha)^* = (\alpha\beta)^* - (\beta\alpha)^* = \beta^*\alpha^* - \alpha^*\beta^* = \beta^\alpha - \alpha\beta$ and if $\alpha \in FG^+$ and $\beta \in FG^-$ then $(\alpha\beta - \beta\alpha)^* = (\alpha\beta)^* - (\beta\alpha)^* = \beta^*\alpha^* - \alpha^*\beta^* = -\beta^\alpha + \alpha\beta$ thus $[FG^+, FG^+] \subseteq FG^-$ and $[FG^-, FG^+] \subseteq FG^+$. Noting that $p^m + 1$ is even we get $(z - z^{-1})\omega = [\alpha_1, ..., \alpha_{p^{m-1}+1}] \in FG^-$, so $((z - z^{-1})\omega)^* = -(z - z^{-1})\omega$ but we also have $((z - z^{-1})\omega)^* = -(z - z^{-1})\omega^*$ because $(z - z^{-1})$ is central and skew. Hence $(z - z^{-1})\omega = (z - z^{-1})\omega^*$ and therefore $(z - z^{-1})\omega = (z - z^{-1})\beta_1$, where $\beta 1 = \frac{\omega + \omega^*}{2}$ is symmetric.

Thus we have $[\alpha_1, ..., \alpha_{p^m+1}] = [(z - z^{-1})\beta_1, \alpha_{p^{m-1}+2}, ..., \alpha_{p^m+1}] = (z - z^{-1})[\beta_1, \alpha_{p^{m-1}+2}, ..., \alpha_{p^m+1}] = ... = (z - z^{-1})^p\beta_p$

Noticing that $(z - z^{-1})^p = z^p - z^{-p} = 1 - 1 = 0$ finishes the proof for $char(F) = p > 2$.

If $char(F) = 0$ we notice that $\mathbb{Z}G^+$ is Lie nilpotent thus so is $(\mathbb{Z}/q\mathbb{Z})G^+$ for any odd prime q. Applying the $char(F) = q$ case to this shows us that $G = Q_8 \times E \times P$, where P is a finite $q - group$. Since this holds for any q, we have $G = Q_8 \times E$. □

2.3 Case: Char(F)=2

In studying the Lie nilpotence of FG^+ or FG^- we have generally assumed that $char(F) \neq 2$. In this section we will give some partial results characterizing G when FG^+ or FG^- is Lie nilpotent and $char(F) = 2$. To begin we should

note that we can simplify the question by noticing that $1 = -1$ because we are working in characteristic 2. This yields the following observation.

Lemma 2.16. *Suppose* $char(F) = 2$, *then* $FG^+ = FG^-$

Proof. If $\alpha \in FG^+$ then $\alpha^* = \alpha$ but since $\alpha + \alpha = 2\alpha = 0$ we have that $\alpha = -\alpha$ thus $\alpha^* = -\alpha$. Similarly if $\alpha^* = -\alpha$ then $\alpha^* = \alpha$ because $\alpha = -\alpha$. Therefore $FG^+ = FG^-$. □

Theorem 2.6. *Suppose* $char(F) = 2$, *and* FG^+ *is Lie nilpotent, then* $G^{2^m} = \{g^{2^m} | g \in G\}$ *is a commutative set for some* m *and* $g^{-1}h^{2^m}g = h^{\pm 2^m}$ *for all* $g, h \in G$.

Proof. Choose n so that 2^n exceeds the index of Lie nilpotence of FG^+. Now for $g, h \in G$ we have $[g + g^{-1}, {}_{2^n} h + h^{-1}] = [g + g^{-1}, h^{2^n} + h^{-2^n}] = 0$ by lemma 1.3 thus $gh^{2^n} + h^{-2^n} + g^{-1}h^{2^n} + g^{-1}h^{-2^n} + h^{2^n}g + h^{-2^n}g + h^{2^n}g^{-1} + h^{-2^n}g^{-1} = 0$, note that I have used $+$ on every term because $char(F) = 2$. Each term in this sum must equal another term in the sum in order to cancel out, thus for gh^{2^n} we have seven possibilities:

(i) $gh^{2^n} = h^{2^n}g$,

(ii) $gh^{2^n} = gh^{-2^n} \Rightarrow h^{2^{n+1}} = 1$ thus $h^{2^{n+1}}g = gh^{2^{n+1}}$,

(iii) $gh^{2^n} = g^{-1}h^{2^n} \Rightarrow g^2 = 1$ thus $h^{2^n}g^2 = g^2h^{2^n}$,

(iv) $gh^{2^n} = g^{-1}h^{-2^n} \Rightarrow g^2h^{2^n} = h^{2^n}g^2 = 1$,

(v) $gh^{2^n} = h^{2^n}g^{-1} \Rightarrow h^{2^n}g = g^{-1}h^{2^n}$ and thus $gh^{2^{n+1}} = h^{2^n}g^{-1}h^{2^n} = h^{2^{n+1}}g$,

(vi) $gh^{2^n} = h^{-2^n}g \Rightarrow h^{2^n}g = gh^{-2^n}$ and thus $g^2h^{2^n} = gh^{-2^n}g = h^{2^n}g^2$,

(vii) $gh^{2^n} = h^{-2^n}g^{-1} \Rightarrow h^{2^n}g = g^{-1}h^{-2^n}$ thus either

$gh^{-2^n} = g^{-1}h^{2^n}$ and thus $g^2h^{2^{n+1}} = h^{2^{n+1}}g^2 = 1$ or

$gh^{-2^n} = h^{-2^n}g$ and then $g = h^{-2^n}gh^{-2^n} = h^{-2^n}g^{-1}h^{-2^n}$ thus $g^2 = 1$ so $g^2h^{2^n} = h^{2^n}g^2$, or

$gh^{-2^n} = h^{2^n}g^{-1}$. In this final case we note that if we switch g and h at the beginning we find that either $g^{2^{n+1}}$ and $h^{2^{n+1}}$ commute or we arrive at this final case and then $hg^{2^n} = g^{-2^n}h$ and $hg^{-2^n} = g^{2^n}h^{-1}$ Thus we find $g^{2^n}h^{2^n} = h^{2^n}g^{-2^n}$ and $g^{2^n}h^{2^n} = h^{-2^n}g^{2^n}$. Thus $g^{2^{n+1}} = h^{2^{n+1}}$ and they commute.

Therefore $G^{2^{n+1}}$ is commutative.

To show that $g^{-1}G^{2^{n+1}}g \subseteq G^{2^{n+1}}$ we again just examine the above cases. In (i), (ii) and (v) there is no problem.

In (iii) we have $g^2 = 1$ so $[g, h^{2^n} + h^{-2^n}] = 0$ and hence (i) or (ii) applies or $gh^{2^n} = h^{-2^n}g$ and thus $g^{-1}h^{-2^n}g = h^{2^n}$.

In (iv) $gh^{2^n} = g^{-1}h^{-2^n}$ and thus $h^{2^{n+1}} = g^{-2}$ hence $g^{-1}h^{2^{n+1}}g = g^{-1}g^{-2}g = g^{-2} = h^{2^{n+1}}$

In (vi) $gh^{2^n} = h^{-2^n}g$ thus $g^{-1}h^{-2^n}g = h^{2^n}$.

In (vii) we have $gh^{2^n}g = h^{-2^n}$ and one of three possibilities:

if $gh^{-2^n} = h^{-2^n}g$ then $g^{-1}h^{2^n}g = h^{2^n}$,

if $gh^{-2^n} = h^{2^n}g^{-1}$ then $h^{-2^n} = g^{-1}h^{2^n}g^{-1}$ thus $h^{-2^{n+1}} = g^{-1}h^{2^n}g^{-1}gh^{2^n}g = g^{-1}h^{2^{n+1}}g$,

and if $gh^{-2^n} = g^{-1}h^{2^n}$ then $g^2 = h^{2^{n+1}}$ hence $g^{-1}h^{2^{n+1}}g = g^{-1}g^2g = g^2 = h^{2^{n+1}}$.

Thus $g^{-1}h^{2^{n+1}}g = h^{\pm 2^{n+1}}$ in each case so $g^{-1}G^{2^{n+1}}g \subseteq G^{2^{n+1}}$. □

Corollary 2.4. *Suppose $char(F) = 2$, and FG^+ is Lie nilpotent, then $\langle G^{2^m} \rangle$ is a normal abelian subgroup of G for some $m \geq 0$.*

Proof. This is an abelian subgroup because it is generated by a commutative subset of G. Given any element $g \in \langle G^{2^m} \rangle$, it is of the form $g = g_1^{2^m} ... g_n^{2^m}$. Calculating we obtain $h^{-1}gh = h^{-1}g_1^{2^m}hh^{-1}g_2^{2^m}hh^{-1}...g_n^{2^m}h = g_1^{\pm 2^m} ... g_n^{\pm 2^m}$. Thus $h^{-1}gh \in \langle G^{2^m} \rangle$. □

Remark 2.6. ([14] Corollary 5.3.10.) Let K be a field of characteristic $p > 0$. The KG satisfies a polynomial identity if and only if G has a p-abelian subgroup of finite index.

Theorem 2.7. *Suppose $char(F) = 2$, and G contains no 2-elements. If FG^+ is Lie nilpotent, then the set of torsion elements $T(G)$ in G is a normal abelian subgroup of G and G has a normal abelian subgroup A of finite index with $T(G) \subseteq A$.*

Proof. Consider any $g \in T(G)$, then $o(g) = m$ for some odd m, and $h = g^{(m+1)/2}$ has $h^2 = g$. Thus $T(G)^2 = T(G)$ and finite induction gives $T(G) = T(G)^{2^n} \subseteq G^{2^n}$ for any fixed n, and thus $T(G)$ is abelian from the previous theorem. If $x, y \in T(G)$ and $k = o(x)o(y)$ then $(xy)^k = x^k y^k = 1$ because $T(G)$ is abelian, thus $T(G)$ is a subgroup. Normality is clear because $o(g) = o(h^{-1}gh)$ for any $g, h \in G$.

Since FG^+ is Lie nilpotent FG satisfies a *-polynomial identity, and thus it satisfies a polynomial identity by remark 2.1. Hence G contains a 2-abelian subgroup of finite index by the previous remark. Since G contains no 2-elements this means that it contains an abelian subgroup A of finite index, say, $[G : A] = m$. This implies that for any $g \in G$ we have $g^m \in A$. Note that since A is an abelian subgroup A^{2^n} is abelian, and it also forms a subgroup, and since $gh^{2^n}g^{-1} = h^{\pm 2^n}$ by theorem 2.6 A^{2^n} is normal. Since for any $g \in G$ we have $g^m \in A$, hence $g^{2^n m} \in A^{2^n}$, thus G has a normal abelian subgroup of finite index. Notice that if $g \in T(G)$ then $h^2gh^{-2} = g$ because $hgh^{-1} = g^{\pm 1}$ for any $g \in T(G) \subseteq G^{2^n}$, thus $aga^{-1} = g$ for all $a \in A^{2^n}$. Thus we can conclude that $\langle A^{2^n}, g\rangle$ is abelian, and since $g \in G^{2^n}$ implies $hgh^{-1} = g^{\pm 1}$ and $\langle A^{2^n}, g\rangle \subseteq G^{2^n}$ we can conclude that $\langle A^{2^n}, g\rangle$ is normal in G. Thus we can assume $T(G) \subseteq A^{2^n}$. □

Remark 2.7. Notice that the above corollary implies if G is torsion with no 2-elements and FG^+ is Lie nilpotent for some field F with $char(F) = 2$ then G is abelian.

Chapter 3

3.1 Lie Nilpotence of FG^-

We now return to the question of characterizing G when $char(F) \neq 2$ and FG^- is Lie nilpotent. The following results can be found in [5].

Lemma 3.1. *Suppose $char(F) \neq 2$ and G contains an abelian subgroup A of index 2 and write $G = A \bigcup xA$, with $x \in G \setminus A$. If $A^2 = 1$ or $(xA)^2 = 1$, then $[FG^-, FG^-] = 0$.*

Proof. We need to prove that $[g - g^{-1}, h - h^{-1}] = 0$ because FG^- is generated by $\{g - g^{-1} | g \in G\}$ over F. Assume $A^2 = 1$, if g or h was in A it would have order 2 and then $g - g^{-1} = 0$ or $h - h^{-1} = 0$. So assume $g, h \in xA$, then we can write $g = xa$, $h = xb$ with $a, b \in A$. Then we have,

$[xa - (xa)^{-1}, xb - (xb)^{-1}] = xaxb - xab^{-1}x^{-1} - a^{-1}x^{-1}xb + a^{-1}x^{-1}b^{-1}x^{-1} - xbxa + xba^{-1}x^{-1} + b^{-1}x^{-1}xa - b^{-1}x^{-1}a^{-1}x^{-1}$

$= xaxb - xabx^{-1} - ab + ax^{-1}bx^{-1} - xbxa + xbax^{-1} + ba - bx^{-1}ax^{-1}$

because a and b have order 2. Noting that $ab = ba$ this reduces to $xaxb + ax^{-1}bx^{-1} - xbxa - bx^{-1}ax^{-1}$. We can further reduce this by noticing $ax = xa'$ and $bx = xb'$ with $a', b' \in A$ because $ax, bx \notin A$ and again noting that a' and b' have order 2 we get,

$[xa - (xa)^{-1}, xb - (xb)^{-1}] = x^2a'b + ab'x^{-2} - x^2b'a - ba'x^{-2}$

Finally $x^2 \in A$ since A has index 2, and so x^2 commutes with a, a', b, and b' and $x^2 = x^{-2}$ thus we get $[g - g^{-1}, h - h^{-1}] = [xa - (xa)^{-1}, xb - (xb)^{-1}] = 0$.

If $(xA)^2 = 1$ then again we can assume $g, h \in A$ for else $g - g^{-1} = 0$ or $h - h^{-1} = 0$, but if $g, h \in A$ then $gh = hg$ and thus $[g - g^{-1}, h - h^{-1}] = 0$ $\square$

Lemma 3.2. *Assume* $[g - g^{-1}, h^{p^n} - h^{-p^n}] = 0$ *for all* $g, h \in G$ *with* $p \neq 2$. *Then for any fixed* $g, h \in G$ *with* $g^2 \neq 1$ *and* $h^{2p^n} \neq 1$ *either* $gh^{p^n} = h^{p^n}g$ *or* $g^4 = h^4 = 1$

Proof. Assume $gh^{p^n} \neq h^{p^n}g$. Expanding the Lie brackets we obtain $[g - g^{-1}, h^{p^n} - h^{-p^n}] = gh^{p^n} - gh^{-p^n} - g^{-1}h^{p^n} + g^{-1}h^{-p^n} - h^{p^n}g + h^{p^n}g^{-1} + h^{-p^n}g - h^{-p^n}g^{-1} = 0$.

Thus gh^{p^n} must equal one of the negative or 2 of the added terms if $char(F) = 3$ as usual. Since $g^2 \neq 1$, $h^{2p^n} \neq 1$ we know that $gh^{p^n} \neq gh^{-p^n}$ and $gh^{p^n} \neq g^{-1}h^{p^n}$.

If $gh^{p^n} = g^{-1}h^{-p^n} = h^{-p^n}g$ then inverting each of these gives $h^{-p^n}g^{-1} = h^{p^n}g = g^{-1}h^{p^n}$ which leaves $hg^{-p^n} = gh^{-p^n}$ thus $g^{-1} = h^{-p^n}gh^{-p^n} = gh^{p^n}h^{-p^n} = g$ thus $o(g) = 2$, a contradiction. If $gh^{p^n} = g^{-1}h^{-p^n} = h^{p^n}g^{-1}$ then $h^{-p^n}g^{-1} = h^{p^n}g = gh^{-p^n}$ which leaves $g^{-1}h^{p^n} = h^{-p^n}g$ and thus $g = h^{p^n}g^{-1}h^{p^n} = gh^{2p^n}$ and $o(h^{p^n}) = 2$ a contradiction. Finally if $gh^{p^n} = h^{p^n}g^{-1} = h^{-p^n}g$ then $h^{-p^n}g^{-1} = gh^{-p^n} = g^{-1}h^{p^n}$ which leaves $g^{-1}h^{-p^n} = hg$. This last equality implies $g = h^{-p^n}g^{-1}h^{-p^n}$, while $gh^{p^n} = h^{-p^n}g$ implies $g = h^{-p^n}gh^{-p^n}$ and together these give $g = g^{-1}$ and thus $g^2 = 1$. Therefore gh^{p^n} cannot equal 2 of the added terms and thus $gh^{p^n} = h^{-p^n}g^{-1}$. This yields $g^{-1} = h^{p^n}gh^{p^n}$. We are now left with 4 terms in that need to cancel out. Since $gh^{p^n} \neq h^{p^n}g$ and g and h^{p^n} do not have order 2 we are left with $gh^{-p^n} = h^{p^n}g^{-1}$ and $g^{-1}h^{p^n} = h^{-p^n}g$ which yield $g = h^{p^n}g^{-1}h^{p^n}$.

If $g^2 \neq g^{-2}$ then $g^2 = g(h^{p^n}g^{-1}h^{p^n}) \neq (h^{p^n}gh^{p^n})g^{-1} = g^{-2}$. Thus h^{p^n} does not commute with $gh^{p^n}g^{-1}$ and as $(gh^{p^n}g^{-1})^2 = gh^{2p^n}g^{-1} \neq 1$ we can redo all the previous calculations with g replaced by $gh^{p^n}g^{-1}$. Therefore $gh^{p^n}g^{-1}h^{p^n} = h^{-p^n}gh^{-p^n}g^{-1}$ and hence $g(h^{p^n}g^{-1}h^{p^n})g(h^{p^n}g^{-1}h^{p^n}) = g^4 = 1$ a contradiction therefore $g^4 = 1$. Now since $h^{2p^n} \neq 1$ then $h^2 \neq 1$ and $g^{p^n} = g$ or g^{-1} so $g^{2p^n} = g^2 \neq 1$. And since $gh^{p^n} \neq h^{p^n}g$ we know that $gh \neq hg$ and $g^{-1}h \neq hg^{-1}$ therefore $g^{p^n}h \neq hg^{p^n}$. Therefore we can go through all the previous arguments

with g and h reversed, and we find $h^4 = 1$. □

Corollary 3.1. *Assume FG semiprime and $char(F) \neq 2$ and FG^- is Lie nilpotent. Then for $g, h \in G$ $gh = hg$ or $g^4 = h^4 = 1$.*

Proof. Since FG is semiprime theorem 2.1 says $[FG^-, FG^-] = 0$ and hence FG^- is commutative. Suppose there exist $g, h \in G$ with $g^2 \neq 1$, $h^2 \neq 1$ and $gh \neq hg$. Then by the previous lemma with $n = 0$ we have $g^4 = h^4 = 1$. □

Definition 3.1. An field F is called nonabsolute if it is not algebraic over a finite field.

Remark 3.1. ([5] Corollary 1.) Let R be a finite dimensional semisimple algebra over a nonabsolute algebraically closed field F with $char(F) \neq 2$. Then the unitary unite group $Un(R)$ does not contain a free group of rank 2 if and only if R^- is commutative.

Remark 3.2. ([8] Lemma 1.1 part (*iii*).) Let R be a K-algebra with involution. Let $S = R/I$ where I is a *-stable nil ideal of R. If $Un(R)$ is 2-related then so is $Un(S)$, furthermore the converse holds if $char(K) > 2$.

Remark 3.3. ([8] Theorem 1.3.) Let G be a finite group and let K be a nonabsolute field of characteristic$\neq$ 2. The unitary unit group $Un(KG)$ is 2-related if and only if

(*i*) G has normal Sylow p-subgroup P with $p = char(K)$, and we set $\overline{G} = G/P$. By convention we set $P = 1$ if $char(K) = 0$, and;

(*ii*) Either $\overline{G}$ is abelian or it has an abelian subgroup $\overline{A}$ of index 2. Furthermore, if the latter occurs, then either $\overline{G} = \overline{A} \bowtie \langle \overline{y} \rangle$ is dihedral or $\overline{A}$ is an elementary abelian 2-group.

Lemma 3.3. *If G is finite and FG is semiprime and FG^- is Lie nilpotent then G is either abelian or dihedral or contains an elementary abelian 2-subgroup of index 2.*

Proof. If F is nonabsolute then we can take some t which is transcendental over F and then let K be the algebraic closure of $F(t)$. We know that KG^-

is Lie nilpotent because it KG^- is generated by $\{g - g^- | g \in G\}$ over K and this set is Lie nilpotent because FG^- is Lie nilpotent.

Let J be the jacobson radical of KG, then KG/J is semiprime. Suppose $x + J \in KG/J$ is skew, then $x + J = -x^* + J$, so $x - x^* + J = 2x + J$ thus $x + J = x/2 - (x/2)^* + J$. Therefore any skew element of KG/J is of the form $y + J = x - x^* + J$ and it is therefore the homomorphic image of a skew element in KG, thus $(KG/J)^-$ is Lie nilpotent, and thus commutative by remark 2.3. This implies that the unitary unit group $Un(KG/J) = \{x \in (KG/J) | xx^* = 1\}$ does not contain a free group of rank 2 by remark 3.1. This implies that $Un(KG)$ does not contain a free group of rank 2, by remark 3.2, and hence by remark 3.3 G has a normal Sylow p-subgroup P with G/P satisfying the conclusion. But since FG is semiprime, G contains no nontrivial finite normal p-subgroups, and since G is finite, this implies that $P = 1$. □

Lemma 3.4. *Assume FG semiprime and $char(F) \neq 2$ and FG^- is Lie nilpotent. Then either,*

(i) $G^4 = 1$ *or,*

(ii) $A = \{g \in G | o(g) \neq 2\}$ *is a normal abelian subgroup of G and $(G \setminus A)^2 = 1$.*

Proof. If (ii) does not hold then there exist $g, h \in$G with $g^2 \neq 1$, $h^2 \neq 1$ and $gh \neq hg$, but then $g^4 = h^4 = 1$ by corollary 3.1. Thus we have left to show that for any arbitrary $a \in G$, $a^4 = 1$.

Assume $a^4 \neq 1$ then $ga = ag$ and $ha = ah$ for otherwise applying corollary 3.1 would give $a^4 = 1$. Thus $a \in Z(\langle g, h, a \rangle)$. If $o(h) = \infty$ then corollary 2.1 implies $F\langle g, h, a \rangle$ is Lie nilpotent of index 2, thus $\langle g, h, a \rangle$ is abelian contrary to assumption that $gh \neq hg$. Hence $\langle g, h, a \rangle$ is finite. Therefore by lemma 3.3 $\langle g, h, a \rangle$ is either dihedral or contains an elementary abelian 2-subgroup of index 2. If it is dihedral we would have all elements of order 4 commute but $o(g) = o(h) = 4$ and $gh \neq hg$. Thus $\langle g, h, a \rangle$ contains an elementary abelian 2-subgroup A of index 2. Hence $a^2 \in A$ thus $a^4 = 1$, this contradiction concludes the proof. □

Lemma 3.5. *If FG^- is Lie nilpotent and G is torsion, then G is locally finite.*

Proof. FG satisfies a *-polynomial identity, and hence a polynomial identity by remark 2.1. Thus the FC subgroup ϕ is of finite index, and $|\phi'| < \infty$ by theorem 1.1. Therefore $|(G/\phi')/(\phi/\phi')| = |G/\phi| < \infty$, hence ϕ/ϕ' is an abelian subgroup of finite index in G/ϕ' which is torsion, thus G/ϕ' is locally finite. This implies that G is locally finite as ϕ' is finite. □

Theorem 3.1. *Assume FG semiprime and $char(F) \neq 2$ then FG^- is Lie nilpotent if and only if one of the following occurs:*

(i) *G is abelian;*

(ii) *$A = \{g \in G | o(g) \neq 2\}$ is a normal abelian subgroup of G and $(G \backslash A)^2 = 1$;*

(iii) *G contains an elementary abelian 2-group of index 2.*

Proof. It follows from theorem 2.1 that FG^- is Lie nilpotent if and only if $[FG^-, FG^-] = 0$. If G is abelian then FG is Lie nilpotent and so FG^- is Lie nilpotent. If (ii) holds then $g - g^{-1} = 0$ unless $g \in A$, but if $g, h \in$A then they commute so $[g - g^{-1}, h - h^{-1}] = 0$ and hence $[FG^-, FG^-] = 0$. If (iii) holds then lemma 3.1 implies $[FG^-, FG^-] = 0$.

Conversely assume $[FG^-, FG^-] = 0$ and G is not abelian. If $G^4 \neq 1$ then we are in case (ii) by lemma 3.4, similarly is $G^4 = 1$ and all elements of order 4 commute then we are in case (ii). Thus we can assume that there exist $g, h \in G$ with $o(g) = o(h) = 4$ and $gh \neq hg$. Let $H = \langle w, x, y, z \rangle$ with $o(w) = o(x) = o(y) = o(z) = 4$ then H is finite by lemma 3.5 and hence it is either dihedral in which case all elements of order 4 commute, or it contains an elementary abelian 2-subgroup A of index 2. Since w, x, y, z all have order 4, we have that $w^{-1}A = xA$ thus $wx \in A$ and in the same way $yz \in A$ thus $B = \langle xy | x, y \in G, o(x) = o(y) = 4 \rangle$ is abelian. Doing the above calculations with $y = g$ and $z = h$ forces H to not be dihedral and thus we get that $wx \in A$ thus $(wx)^2 = 1$. Therefore B is an elementary abelian 2-subgroup of G.

Let A be a maximal elementary abelian 2-subgroup of G containing B. Choose $x \in G \setminus A$ and suppose $o(x) = 2$. If $xa = ax$ for all $a \in A$ then $\langle A, x\rangle \supset A$ and yet is still elementary abelian, contrary to maximality of A. Thus $ax \neq xa$ for some $a \in$A. Then $xaA = xA$ and $(xa)^2 \neq 1$ for else $xa = ax$ since $o(a) = o(x) = 2$ so $o(xa) = 4$. Thus any coset of A can be written $x'A$ with $o(x') = 4$.

Suppose xA and yA are cosets of A, with $o(x) = o(y) = 4$, then $xy^{-1} \in B \subseteq A$, thus $xA = yA$. Therefore A has index 2. □

Theorem 3.2. *Assume $char(F) = p > 2$, G is torsion, and FG^- is Lie nilpotent. Then $P = \{g \in G | o(g) = p^n, n \geq 0\}$ is a normal subgroup of G and letting $\overline{G} = G/P$ we have:*

(i) FP is Lie nilpotent and

(ii) $F\overline{G}$ is semiprime and $[F\overline{G}^-, F\overline{G}^-] = 0$.

Proof. Given $p_1, p_2 \in P$, $\langle p_1, p_2\rangle$ is finite since G is locally finite from lemma 3.5. Any finite group generated by p-elements is a p-group, hence $p_1p_2 \in P$. This shows that P is a subgroup of G, and since FP^- is Lie nilpotent, FP is Lie nilpotent by theorem 2.3.

Note that P is a normal subgroup of G, since for any p-element p, $o(g^{-1}pg) = o(p)$. Now to prove (ii) all we need to show is that $F\overline{G}$ is semiprime and theorem 3.1 completes the proof. But $\overline{G}$ is torsion and contains no p-elements, therefore $F\overline{G}$ is semiprime by theorem 1.2. □

Theorem 3.3. *Assume $char(F) = p > 2$, and G is not torsion. FG^- is Lie Nilpotent if and only if there exists a normal subgroup H of G with H nilpotent, p-abelian and $(G \setminus H)^2 = 1$.*

Proof. Let $H = \langle g \in G | o(g) \neq 2\rangle$ be the subgroup of G generated by all elements with orders different from 2. H is clearly normal because given any generator h of H, then $o(g^{-1}hg) = o(h) \neq 2$. Since G is not torsion it contains an element x of infinite order, and since $o(x) \neq 2$ then $x \in H$.

Since FG^- is Lie nilpotent $0 = [g - g^{-1}, {}_{p^n} h - h^{-1}] = [g - g^{-1}, h^{p^n} - h^{-p^n}]$ by lemma 1.3. for all $g, h \in G$. Thus if $g^2 \neq 1$ and $h = x$ lemma 3.2 says

that since $x^4 \neq 1$ then $gx^{p^n} = x^{p^n}g$. This implies that x^{p^n} commutes with a generating set for H and thus $x^{p^n} \in Z(H)$. Since $o(x^{p^n}) = \infty$ we have $|Z(H)^2| = \infty$ and FH^- is Lie nilpotent therefore FH is Lie nilpotent by corollary 2.2. Application of theorem 1.4 then implies that H is nilpotent and p-abelian, and $(G \setminus H)^2 = 1$ by construction.

Conversely suppose G contains a normal subgroup of H which is nilpotent and p-abelian and $(G \setminus H)^2 = 1$. FG^- is generated by $\{g - g^{-1} | g \in G\}$ over F, and FH^{-1} is generated by $\{h - h^{-1} | h \in H\}$ over F, but these generating sets are the same because for $g \in G \setminus H$ we have $g - g^{-1} = g^{-1}(g^2 - 1) = g^{-1}(0) = 0$, thus $FG^{-1} = FH^-$. Therefore we need only show that FH^- is Lie nilpotent, but this is clear by theorem 1.4. □

3.2 Conclusions

Thus we have in chapter one a characterization of when a group algebra is Lie nilpotent or Lie solvable and in chapters two and three we have described the conditions under which the symmetric or skew symmetric elements in a group algebra are lie nilpotent if the characteristic of the field is not two. Future work could be done in studying when FG^+ or FG^- is Lie solvable, as we have few results in that area, or in studying the characteristic 2 case further when FG^+ is Lie nilpotent or Lie solvable.

Bibliography

[1] Amitsur, S.A., Rings with Involution, Israel Journal of Mathematics, 6, (1968), 99-106.

[2] Amitsur, S.A., Identities in Rings with Involution, Israel Journal of Mathematics, 7, (1968), 63-68.

[3] Giambruno, A., Algebraic Conditions on Rings with Involution, Journal of Algebra, 50, (1978), 190-212.

[4] Giambruno, A. and Polcino Milies, C., Lie Nilpotence in Group Algebras, preprint, (2002).

[5] Giambruno, A, and Polcino Milies, C., Unitary Units and Skew Elements in Group Algebras, Manuscripta Mathematica, 111, (2003), 195-209.

[6] Giambruno, A. and Sehgal, S.K., A Lie Property in Group Rings, Proceedings of the American Mathematical Society, 105, (1989), 287-292.

[7] Giambruno, A. and Sehgal, S.K., Lie Nilpotnence of Group Rings, Communications in Algebra, 21, (1993), 4253-4261.

[8] Goncalves, J.Z., and Passman, D.S., Unitary Units in Group Algebras, Isreal Journal of Mathematics, 125, (2001), 131-155.

[9] Hall, M.,The Theory of Groups, Macmillan, New York, (1959).

[10] Herstein, I.N., Non-Commutative Rings, Mathematical Assotiation of America, Washington, DC, (1968).

[11] Herstein, I.N., Rings with Involution, University of Chicago Press, Chicago, (1976).

[12] Lee, G.T$_{\pi}$ Group Rings whose Symmetric Elements are Lie Nilpotent, Proceedings of the American Mathematical Society, 127, (1999), 3153-3159.

[13] Passi, I.B.S., Passman, D.S. and Sehgal, S.K., Lie Solvable Group Rings, The Canadian Journal of Mathematics, 25, (1973), 748-757.

[14] Passman, Donald S., The Algebraic Structure of Group Rings, Robert E. Krieger Publishing Company, Inc., Malabra, Florida, (1985).

[15] Polcino Milies, C. and Sehgal, S.K., An Introduction to Group Rings, Algebras and Applications, 1, Kluwer Academic Publishers, Dordrecht, Netherlands, (2002).

[16] Sehgal, S.K., Topics in Group Rings, Monographs and Textbooks in Pure and Applied Mathematics, 50, Marcel Dekker Inc., New York, (1978).